ME 115(b): Homework #2 Solution

Problem #2: (Prob. 10 of MLS Chapt. 3). As stated in the textbook, this problem does not make sense unless the manipulator is "non-redundant." Let's consider the more general case (discussed in class) where the *manipulability measure* is defined as

$$manipulability = \sqrt{\det(J(\theta)J^T(\theta))}$$

where $J(\theta)$ is the manipulator Jacobian. Using the singular value decomposition, recall that the $m \times n$ Jacobian matrix can be expressed as $J = U\Sigma V^T$, where U and V are respectively $m \times m$ and $n \times n$ orthogonal matrices. Σ is a $m \times n$ matrix, with m "singular values" along the diagonal. We will generally assume that $m \leq n$. Thus:

$$\begin{aligned}
\sqrt{det(JJ^T)} &= \sqrt{det(U\Sigma V^T V\Sigma^T U^T)} = \sqrt{det(U\Sigma\Sigma^T U^T)} \\
&= \sqrt{det(U)det(\Sigma\Sigma^T)det(U)} = \sqrt{det(\Sigma\Sigma^T)} \\
&= \sqrt{\sigma_1^2 \cdots \sigma_m^2} = \sigma_1\sigma_2\cdots\sigma_m
\end{aligned}$$

When $m = n$, this measure is equivalent to the measure discussed in the text.

Problem #3

Part (a): Let $\theta_1, \theta_2, \theta_3$ be associated with the B, A, C

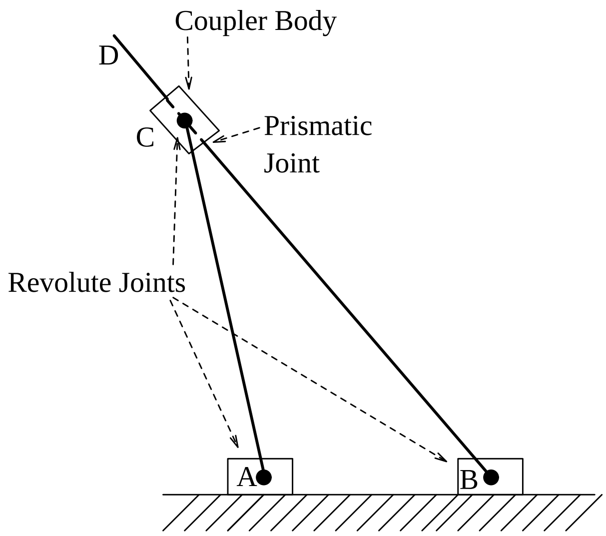

Figure 1: Mechanism

The twist coordinates for revolute joints A and B are straightforward:

$$\begin{aligned}
\xi_B &= [0\ 0\ 0\ 0\ 0\ -1]^T \\
\xi_A &= [0\ -1\ 0\ 0\ 0\ -1]^T
\end{aligned} \tag{1}$$

and Joint C is

$$\begin{aligned}
v &= -\omega \times q \\
where & \\
\omega &= [0,0,-1]^T \\
v &= [-1 - l_2\cos\theta_2,\ l_2\sin\theta_2,\ 0]^T \\
so & \\
\xi_C &= [-l_2\sin\theta_2,\ -1 - l_2\cos\theta_2,\ 0,\ 0,\ 0,\ -1]^T
\end{aligned} \tag{2}$$

Joint D is prismatic, and therefore an infinite pitch joint. Its screw coordinates are simply:

$$\xi_4 = [-\cos\theta_1,\ \sin\theta_2,\ 0,\ 0,\ 0,\ 0]^T \tag{3}$$

Part (b):

A stationary configuration of joint A will occur when:

$$det \begin{bmatrix} \xi_B \cdot \xi_B & \xi_B \cdot \xi_C & \xi_B \cdot \xi_D \\ \xi_C \cdot \xi_B & \xi_C \cdot \xi_C & \xi_C \cdot \xi_D \\ \xi_D \cdot \xi_B & \xi_D \cdot \xi_C & \xi_D \cdot \xi_D \end{bmatrix} = 0 \tag{4}$$

For this linkage:

$$det \begin{bmatrix} 2 & 2 + l_2\cos\theta_2 & -\sin\theta_1 \\ 2 + l_2\cos\theta_2 & l_2^2 + 2l_2\cos\theta_2 + 2 & -l_2\sin(\theta_2 - \theta_1) - \sin\theta_1 \\ -\sin\theta_1 & -l_2\sin(\theta_2 - \theta_1) - \sin\theta_1 & 1 \end{bmatrix} = l_2^2\cos^2(\theta_1 - \theta_2) \tag{5}$$

The solutions to this equation represent the singular solutions.

Problem #4 (MLS problem 3.21)

Part (a): We can think of this mechanism as consisting of 8 bodies (noticing that each prismatic joint can be thought of as two bodies). The total number of *relative* degrees of freedom possed by this collection of bodies is 3(8-1) = 21. This collection of bodies is held together by 6 revolute joints and 3 prismatic joints. Thus, the number of constraints is $9 \cdot 2 = 18$. Therefore, the mechanism as 21-18=3 internal degrees of freedom.

Part (b): The structure equations can be derived either using the **P**roduct **Of** **E**xponentials (POE) method, or the Denavit-Hartenberg (DH) convention. Let's consider the POE approach. In this approach, the structure equations take the form:

$$e^{\xi_{11}\alpha_1} e^{\xi_{12} d_1} e^{\xi_{13}\beta_1} g_{bt}(0) = e^{\xi_{21}\alpha_2} e^{\xi_{22} d_2} e^{\xi_{23}\beta_2} g_{bt}(0) = e^{\xi_{31}\alpha_3} e^{\xi_{32} d_3} e^{\xi_{33}\beta_3} g_{bt}(0)$$

Using the obvious "home position," the twists are:

$$\begin{aligned}
\xi_{11} &= \begin{bmatrix} h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \xi_{13} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{21} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \xi_{23} = \begin{bmatrix} -h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{31} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{32} = \frac{1}{\sqrt{w^2+b^2}} \begin{bmatrix} w \\ h \\ 0 \end{bmatrix}; \quad \xi_{33} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix}
\end{aligned} \tag{6}$$

Note that the angles α_i and β_i can be found using the law of cosines. For example:

$$\cos\alpha_1 = \frac{h^2 + d_1^2 - d_3^2}{2hd_1}$$

Part (c): Using the notation in the figure above, we can see that the location of the points $P_1, \cdots, P_4$ are:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \quad P_2 = \begin{bmatrix} 0 \\ -h/2 \end{bmatrix}; \\
P_3 &= \begin{bmatrix} x \\ y \end{bmatrix}; \quad +R_\phi \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \quad P_4 = \begin{bmatrix} x \\ y \end{bmatrix}; \quad +R_\phi \begin{bmatrix} 0 \\ -h/2 \end{bmatrix};
\end{aligned} \tag{7}$$

The actuator lengths can be derived as:

$$\begin{aligned}
d_1 &= ||P_3 - P_1|| \\
d_2 &= ||P_4 - P_2|| \\
d_3 &= ||P_3 - P_2||
\end{aligned} \tag{8}$$

Part (d):

The spatial Jacobians are simply:

$$[\,\xi_{11} \quad \xi_{12}' \quad \xi_{13}'\,] \begin{bmatrix} \dot\alpha_1 \\ \dot d_1 \\ \dot\beta_1 \end{bmatrix} = [\,\xi_{21} \quad \xi_{22}' \quad \xi_{23}'\,] \begin{bmatrix} \dot\alpha_2 \\ \dot d_2 \\ \dot\beta_2 \end{bmatrix} = [\,\xi_{31} \quad \xi_{32}' \quad \xi_{33}'\,] \begin{bmatrix} \dot\alpha_3 \\ \dot d_3 \\ \dot\beta_3 \end{bmatrix}$$

where ξ_{ij}' is the appropriately transformed twist coordinates. In this case, the transformed twists can be determined by inspection:

$$
\begin{aligned}
\xi'_{12} &= \begin{bmatrix} \cos\alpha_1 \\ \sin\alpha_1 \\ 0 \end{bmatrix}; \qquad \xi'_{13} = \begin{bmatrix} (w+d_1)\sin\alpha_1 + h/2 \\ -(w+d_1)\cos\alpha_1 \\ 1 \end{bmatrix} \\
\xi'_{22} &= \begin{bmatrix} w\cos\alpha_1 - h\sin\alpha_1 \\ w\sin\alpha_1 + h\cos\alpha_1 \\ 0 \end{bmatrix}; \qquad \xi'_{23} = \begin{bmatrix} (w+d_2)\sin\alpha_2 - h/2 \\ -(w+d_2)\cos\alpha_2 \\ 1 \end{bmatrix} \\
\xi'_{32} &= \begin{bmatrix} \cos\alpha_2 \\ \sin\alpha_2 \\ 0 \end{bmatrix}; \qquad \xi'_{33} = \xi'_{13}
\end{aligned}
\tag{9}
$$