ME 115(a): Solution to Homework #4

Problem 1: (Problem 11(a,b,d,e) in Chapter 2 of MLS).

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi' = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi' = (h\hat{\xi}' h^{-1})^\vee \tag{1}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{\xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.$$

Expanding Eq. (1) gives:

$$\xi = (h\hat{\xi}' h^{-1})^\vee = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^\vee = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$.

Part (d): Let

$$g = \begin{bmatrix} A & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix}$$

where $A \in SO(2)$ and $\vec{p} \in \mathbb{R}^2$. Then direct calculation shows that $\dot{g}g^{-1}$ and $g^{-1}\dot{g}$ are twists. The spatial and body velocities have definitions analogous to those for 3-dimensional rigid bodies

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\vec{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T\vec{p} + R\vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \vec{v}^s = R\vec{v}^b - R\hat{\omega}^b R^T \vec{p} = R\vec{v}^b - \hat{\omega}^s \vec{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

Using this result:

$$\vec{v}^s \;=\; R\vec{v}^b - \hat{\omega}^s \vec{p} = R\vec{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \qquad (2)$$

Therefore:

$$V^s = \begin{bmatrix} \vec{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix} V^b$$

Problem 2: (Problem 14 in Chapter 2 of MLS).

Part (a): Let $g \in SE(3)$ denote a homogeneous transformation matrix:

$$g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Then:

$$g^{-1} = \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_{g^{-1}} = \begin{bmatrix} R^T & -(\widehat{R^T\vec{p}})R^T \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

where we have made use of the identity $(\widehat{R^T\vec{p}}) = R^T\hat{p}R$. Let's now compute $Ad_g Ad_{g^{-1}}$:

$$Ad_g Ad_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Hence, $Ad_{g^{-1}}$ must equal $(Ad_g)^{-1}$ since $Ad_g Ad_{g^{-1}} = I$.

Part (b): If

$$g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Then

$$g_1 g_2 = \begin{bmatrix} R_1 R_2 & \vec{p}_1 + R_1\vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence:

$$\begin{aligned} Ad_{g_1 g_2} &= \begin{bmatrix} R_1 R_2 & (\vec{p}_1 + R_1\vec{p}_2)\hat{}\, R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_1^T R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2 R_2 \\ 0 & R_2 \end{bmatrix} = Ad_{g_1} Ad_{g_2} \end{aligned}$$

Problem 3: (Problem 15 in Chapter 2 of MLS).

Since $g_{ac} = g_{ab}g_{bc}$, using the product rule for differentiation yields:

$$\begin{aligned}
\hat{V}^b_{ac} &= g^{-1}_{ac}\dot{g}_{ac} \\
&= g^{-1}_{bc}g^{-1}ab(\dot{g}_{ab}g_{bc} + g_{ab}\dot{g}_{bc}) \\
&= g^{-1}_{bc}\hat{V}^b_{ab}g_{bc} + \hat{V}^b_{bc}
\end{aligned} \tag{3}$$

Converting to vector form via the $\vee$ operator yields:

$$\vec{V}^b_{ac} = (\hat{V}^b_{ac})^\vee = Ad_{g^{-1}_{bc}}\vec{V}^c_{ab} + \vec{V}^b_{bc}$$

Problem 4: (Problem 18(e) in Chapter 2 of MLS.)

Part (e): Let the position and orientation of a moving rigid body be given by $R(t)$ and $\vec{p}(t)$. Let V^b be the body velocity of the rigid body, and let F^b be a wrench applied to the body, expressed in body coordinates. The power applied to the body due to this wrench is given by:

$$V^b \cdot F^b = (V^b)^T F^b \tag{4}$$

Let V^h denote the velocity of the body in hybrid coordinates. Similarly, define the hybrid wrench to be F^h. We will define F^h to be the wrench that preserves the amount of power in Eq. (4):

$$\begin{aligned}
V^b \cdot F^b = (V^b)^T F^b &= V^h \cdot F^h \\
&= (V^h)^T F^h \\
&= (\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} V^b)^T F^h \\
&= (V^b)^T \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h
\end{aligned}$$

Hence, it must be true that:

$$F^b = \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h \qquad \text{or} \qquad F^h = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} F^b$$

Problem 5: Let points P, Q, and R be located at positions $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ with respect to a fixed observer. Let the velocities of each of these points be $\dot{\vec{r}}_1$, $\dot{\vec{r}}_2$, and $\dot{\vec{r}}_3$. Recall that a spatial velocity has the form

$$\hat{V}^s = \begin{bmatrix} \hat{\omega}^s & \vec{v}^s \\ \vec{0}^T & 0 \end{bmatrix}$$

In general, let $\vec{p}$ denote the location of a particle in the moving body, with respect to the observing frame. Recall that the velocity of this particle, $\dot{\vec{p}}$, as seen in the observing frame, is related to the location of the particle (as seen in the stationary observing frame) by

$\dot{\vec{p}} = \hat{V}^s\vec{p}$—where we have used the same notation for homogeneous coordinates. Thus, for each of the particles the following relationship holds:

$$\begin{aligned} \dot{\vec{r}}_1 &= \vec{\omega}^s \times \vec{r}_1 + \vec{v}^s \quad (5) \\ \dot{\vec{r}}_2 &= \vec{\omega}^s \times \vec{r}_2 + \vec{v}^s \quad (6) \\ \dot{\vec{r}}_3 &= \vec{\omega}^s \times \vec{r}_3 + \vec{v}^s \quad (7) \end{aligned}$$

Hence, we have three simultaneous equations in the unknowns $\vec{\omega}^s$ and $\vec{v}^s$. To solve these equations, we follow closely the technique that was used in the handout entitled " Computing the Screw Parameters of a Rigid Body Displacement."

Step 1: subtract Equation 7 from Equations (5) and (6).

$$\begin{aligned} \dot{\vec{r}}_1 - \dot{\vec{r}}_3 &= \vec{\omega}^s \times (\vec{r}_1 - \vec{r}_3) \quad (8) \\ \dot{\vec{r}}_2 - \dot{\vec{r}}_3 &= \vec{\omega}^s \times (\vec{r}_2 - \vec{r}_3) \quad (9) \end{aligned}$$

Step 2: Take the cross product of the vector $(\dot{\vec{r}}_2 - \dot{\vec{r}}_3)$ with Equation (8).

$$(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\dot{\vec{r}}_1 - \dot{\vec{r}}_3) = (\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\vec{\omega}^s \times (\vec{r}_1 - \vec{r}_3)). \quad (10)$$

Step 3: We now use two facts. First, we use the triple vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$. Second, from Equation (9) we know that $\dot{\vec{r}}_2 - \dot{\vec{r}}_3$ is orthogonal to $\vec{\omega}$. Hence, Equation (10) reduces to

$$(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\dot{\vec{r}}_1 - \dot{\vec{r}}_3) = \vec{\omega}[(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \cdot (\vec{r}_1 - \vec{r}_3)].$$

The quantity $\vec{\omega}$ is then easily found as

$$\vec{\omega} = [(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \cdot (\vec{r}_1 - \vec{r}_3)]^{-1}(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\dot{\vec{r}}_1 - \dot{\vec{r}}_3).$$

Step 4: Any one of Equations (5), (6), or (7) could be used to solve for $\vec{v}^s$.