ME 115(a): Solution to Homework #4

Problem 1:

Let the three rotating frames be termed the "ψ-frame," "ϕ-frame," and the "γ-frame." The spatial angular velocity of the body will be the same as the spatial angular velocity of the γ-frame:

$$\vec{\omega}^s = {}^S R^{\psi} \vec{\omega}^b_{\psi} + {}^S R^{\phi} \vec{\omega}^b_{\phi} + {}^S R^{\gamma} \vec{\omega}^b_{\gamma} \tag{1}$$

where $\vec{\omega}^b_{\psi}$ is the body angular velocity of the ψ-frame, ${}^S R^{\psi}$ is the orientation of the ψ-frame with respect to the stationary frame, etc. The body angular velocities are simply:

$$\vec{\omega}^b_{\psi} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \qquad \vec{\omega}^b_{\phi} = \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} \qquad \vec{\omega}^b_{\gamma} = \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix}$$

while the orientations of the frames are:

$${}^S R^{\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{\psi} R^{\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$${}^{\phi} R^{\gamma} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using these frames, one can determine:

$${}^S R^{\phi} = {}^S R^{\psi\psi} R^{\phi} \qquad {}^S R^{\gamma} = {}^S R^{\psi\psi} R^{\phi\phi} R^{\gamma}.$$

Substituting into Equation (1) results in:

$$\vec{\omega}^s = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} -\dot{\phi}\sin\psi \\ \dot{\phi}\cos\psi \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\gamma}\cos\psi\sin\phi \\ \dot{\gamma}\sin\psi\sin\phi \\ \dot{\gamma}\cos\phi \end{bmatrix} = \begin{bmatrix} -\dot{\phi}\sin\psi + \dot{\gamma}\cos\psi\sin\phi \\ \dot{\phi}\cos\gamma + \dot{\gamma}\sin\psi\sin\phi \\ \dot{\psi} + \dot{\gamma}\cos\phi \end{bmatrix} \tag{2}$$

Problem 2: (Problem 11(d,e) in Chapter 2 of MLS).

Part (d): Let

$$g = \begin{bmatrix} A & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix}$$

where $A \in SO(2)$ and $\vec{p} \in \mathbb{R}^2$. Then direct calculation shows that $\dot{g}g^{-1}$ and $g^{-1}\dot{g}$ are twists. The spatial and body velocities have definitions analogous to those for 3-dimensional rigid bodies

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\vec{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T \vec{p} + R\vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \vec{v}^s = R\vec{v}^b - R\hat{\omega}^b R^T \vec{p} = R\vec{v}^b - \hat{\omega}^s \vec{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

Using this result:

$$\vec{v}^s = R\vec{v}^b - \hat{\omega}^s \vec{p} = R\vec{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \tag{3}$$

Therefore:

$$V^s = \begin{bmatrix} \vec{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix} V^b$$

Problem 3: (Problem 14 in Chapter 2 of MLS).

Part (a): Let $g \in SE(3)$ denote a homogeneous transformation matrix:

$$g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Then:

$$g^{-1} = \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_{g^{-1}} = \begin{bmatrix} R^T & -(\widehat{R^T\vec{p}})R^T \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

where we have made use of the identity $(\widehat{R^T\vec{p}}) = R^T\hat{p}R$. Let's now compute $Ad_g Ad_{g^{-1}}$:

$$Ad_g Ad_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Hence, $Ad_{g^{-1}}$ must equal $(Ad_g)^{-1}$ since $Ad_g Ad_{g^{-1}} = I$.

Part (b): If

$$g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Then

$$g_1 g_2 = \begin{bmatrix} R_1 R_2 & \vec{p}_1 + R_1 \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence:

$$\begin{aligned} Ad_{g_1 g_2} &= \begin{bmatrix} R_1 R_2 & (\vec{p}_1 + R_1 \vec{p}_2)\hat{} R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_1^T R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2 R_2 \\ 0 & R_2 \end{bmatrix} = Ad_{g_1} Ad_{g_2} \end{aligned}$$

Problem 4: (Problem 17(a,b) in Chapter 2 of MLS).

Part(a): Let ξ_1 and ξ_2 denote the coordinates of two screws, as described in Frame A. Let frame B be displaced relative to frame A by g. Let η_1 and η_2 be the representation of these screws in Frame B. Hence:

$$\xi_1 = Ad_g \eta_1 \qquad \xi_2 = Ad_g \eta_2$$

Letting $\circ$ denote the reciprocal product:

$$\begin{aligned} \eta_1 \circ \eta_2 &= \eta_1^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \eta_2 \\ &= (Ad_{g^{-1}} \xi_1)^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} (Ad_{g^{-1}} \xi_2) \\ &= \xi_1^T \left(Ad_{g^{-1}}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Ad_{g^{-1}} \right) \xi_2 \end{aligned}$$

A direct calculation shows that

$$Ad_{g^{-1}}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Ad_{g^{-1}} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Hence, $\eta_1 \circ \eta_2 = \xi_1 \circ \xi_2$.

Part(b): Using the same notation as in part (a):

$$\begin{aligned} \eta_1 \cdot \eta_2 &= \eta_1^T \eta_2 \\ &= (Ad_{g^{-1}} \xi_1)^T (Ad_{g^{-1}} \xi_2) \\ &= \xi_1^T Ad_{g^{-1}}^T Ad_{g^{-1}} \xi_2 \\ &= \xi_1^T \begin{bmatrix} I & -\hat{p} \\ \hat{p} & I - \hat{p}^2 \end{bmatrix} \xi_2 \neq \xi_1 \cdot \xi_2 \end{aligned}$$

Problem 5: (Problem 18(b,c,d,e) in Chapter 2 of MLS.)

Part (b): There are many ways to solve this problem. For example, you could either start with Proposition 2.14 or Proposition 2.15 on page 59 of MLS which relate the velocities of three frames, A, B, and C. Let's choose Prop. 2.15:

$$V_{ac}^b = Ad_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b \tag{4}$$

Using the fact that

$$V_{ac}^h = \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V_{ac}^b$$

Eq. (4) can be written as:

$$\begin{aligned} V_{ac}^h &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} (Ad_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b) \\ &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} \begin{bmatrix} R_{bc}^T & -R_{bc}^T \hat{p}_{bc} \\ 0 & R_{bc}^T \end{bmatrix} V_{ab}^b + \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V_{bc}^b \\ &= \begin{bmatrix} R_{ab} & -R_{ab}\hat{p}_{bc} \\ 0 & R_{ab} \end{bmatrix} V_{ab}^b + \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{bc} & 0 \\ 0 & R_{bc} \end{bmatrix} V_{bc}^b \\ &= \begin{bmatrix} I & -(\widehat{R_{ab}p_{bc}}) \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} V_{ab}^b + Ad_{R_{ab}} V_{bc}^h \\ &= Ad_{-R_{ab}p_{bc}} V_{ab}^h + Ad_{R_{ab}} V_{bc}^h \end{aligned} \tag{5}$$

Part (c): Let frames A and B be stationary "spatial" frames, and let Frame C be fixed to a moving body. Let V_{bc}^h be the hybrid velocity of the body, as seen by an observer in the B frame. If we now want to express this velocity as seem by an observer in the A frame (i.e., changing the spatial frame), we need to calculate V_{ac}^h. You can do this using the results of part (b) of this problem (which was not assigned), which derived the result:

$$V_{ac}^h = Ad_{-R_{ab}p_{bc}} V_{ab}^h + Ad_{R_{ab}} V_{bc}^h \tag{6}$$

If you chose this approach, then since A and B are stationary, $V_{ab}^h = 0$. Hence, Eq. (5) takes the form:

$$V_{ac}^h = Ad_{R_{ab}} V_{bc}^h$$

Hence, the hybrid velocity is dependent on the orientation of the spatial frame, but not its position.

Alternatively, if you don't want to rely upon part (b), you can recall that the expression for the hybrid velocity is:

$$V_{ac}^h = \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac}^s \end{bmatrix}$$

Since $\vec{p}_{ac} = \vec{p}_{ab} + R_{ab}\vec{p}_{bc}$, and $\vec{p}_{ab}$ is constant:

$$\dot{\vec{p}}_{ac} = R_{ab}\dot{\vec{p}}_{bc}.$$

Similarly, $\vec{\omega}_{ac} = R_{ab}\vec{\omega}_{bc}$. Hence, V_{ac}^h is dependent of $\vec{p}_{ab}$, but not R_{ab}.

Part (d): Let A be a stationary spatial frame. Let B and C be two different frames attached to a moving body. Let us assume that the velocity of the rigid body is given by V_{ab}^h. If we now switch the location of the body fixed frame from position B to position C, the hybrid velocity of the body is given by V_{ac}^h. Since B and C are both fixed in the body, then $V_{bc}^h = 0$ in Eq. (5). Hence Eq. (5) reduces to:

$$V_{ac}^h = Ad_{-R_{ab}p_{bc}} V_{ab}^h$$

Hence, the hybrid velocity in only dependent on p_{bc}, the position of the body frame, and not on R_{bc}, the orientation of the body fixed frame. Alternatively, you could compute V_{ac}^h in a "brute force" way:

$$\begin{aligned} V_{ac}^h &= \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}(\vec{p}_{ab} + R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ac}R_{ac}^T)^\vee \end{bmatrix} = \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}_{ab}^s R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ab}R_{bc}R_{bc}^T R_{ab}^T)^\vee \end{bmatrix} \\ &= \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}_{ab}^s R_{ab}\vec{p}_{bc}) \\ \vec{\omega}_{ab}^s \end{bmatrix} = Ad_{-R_{ab}p_{bc}} V_{ab}^h \end{aligned}$$

Thus, the result only depends upon $\vec{p}_{bc}$, and not R_{bc}.

Part (e): Let the position and orientation of a moving rigid body be given by $R(t)$ and $\vec{p}(t)$. Let V^b be the body velocity of the rigid body, and let F^b be a wrench applied to the body, expressed in body coordinates. The power applied to the body due to this wrench is given by:

$$V^b \cdot F^b = (V^b)^T F^b \tag{7}$$

Let V^h denote the velocity of the body in hybrid coordinates. Similarly, define the hybrid wrench to be F^h. We will define F^h to be the wrench that preserves the amount of power in Eq. (7):

$$\begin{aligned} V^b \cdot F^b = (V^b)^T F^b &= V^h \cdot F^h \\ &= (V^h)^T F^h \\ &= (\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} V^b)^T F^h \\ &= (V^b)^T \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h \end{aligned}$$

Hence, it must be true that:

$$F^b = \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h \quad \text{or} \quad F^h = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} F^b$$