ME 115(a): Solution to Homework #5

Problem #1: Consider the two screws, S_1 and S_2, shown in Figure 1(a). The screw axis of S_1 is perpendicular to the plane, P, and has zero pitch: $h_1 = 0$. Similarly, the screw axis of S_2 is also perpendicular to the plane. It intersects the plane a distance a from the intersection of S_1, and its pitch is also zero: $h_2 = 0$. Describe the set of all screws whose axes lie in P and that are reciprocal to both S_1 and S_2.

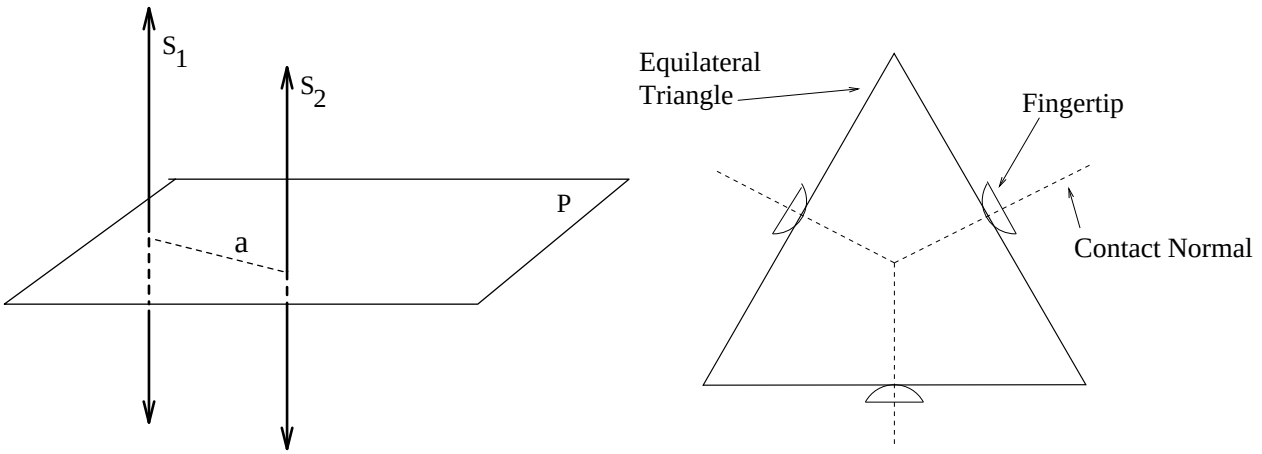

Figure 1: (a) Two Screws; (b) grasp of a triangle

Assign a coordinate system whose origin is located at a point C, where the screw axis of S_1 intersects the horizontal plane, P. Let the z-axis be collinear with the positive S_1 direction, and let the x-axis be collinear with the mutually perpendicular line between S_1 and S_2. In this coordinate system, the screw coordinates of S_1 and S_2 are (recalling that $h_1 = h_2 = 0$ and that the screw axes are separated by a distance a):

$$\xi_1 = \begin{bmatrix} h_1\vec{\omega}_1 + \vec{\rho}_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} h_2\vec{\omega}_2 + \vec{\rho}_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -a \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We require that any screw, S_R, which is reciprocal to both S_1 and S_2 also lie in the plane P. Let's divide the analysis into those screws that have finite pitch and those that potentially have infinite pitch.

We can parametrize all finite pitch screws that lie in P as follows:

$$\xi_R = \begin{bmatrix} h_R\vec{\omega}_R + \vec{\rho_R} \times \vec{\omega}_R \\ \vec{\omega}_R \end{bmatrix}$$

where h_R is the pitch of the reciprocal screw while $\vec{\omega}_R$ is a unit length vector collinear with the screw axis of the reciprocal screw, $\vec{\rho}_R$ is a vector from the origin of the reference frame

described above to a point on the reciprocal screw axis. By assumption, both $\vec{\omega}_R$ and $\vec{\rho}_R$ must also lie in P. We can describe any screw that lies in the plane by two scalars: d (the distance along the mutually perpendicular line between S_1 and S_R) and θ, the angle between the mutually perpendicular line and the x-axis of the reference coordinate system that was defined above. In terms of these scalars:

$$\vec{\omega}_R = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad \vec{\rho}_R = d \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and hence:

$$\xi_R = \begin{bmatrix} -h_R \sin\theta \\ h_R \cos\theta \\ d \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

If S_R is reciprocal to S_1, then the reciprocal product between these two screws must be zero. Letting $\circ$ denote the reciprocal product,

$$\xi_1 \circ \xi_R = d = 0$$

This implies that the screw axis of S_R must intersect the axis of S_1. If S_R is reciprocal to S_2, then:

$$-a\cos\theta = 0.$$

Since $a \neq 0$ by assumption, it must be true that $\cos\theta = 0$, or $\theta = \pm\pi/2$. This implies that the axis of the reciprocal screw is parallel to x-axis of the coordinate system. Hence, all finite pitch reciprocal screws must simultaneously intersect the axes of S_1 and S_2. They can have any pitch value.

Note that if the pitch of the reciprocal screw is infinite, then it is automatically reciprocal to both screws.

Problem #2:

Question: Is the triangle in Fig. 1(b) immobilized? That is, are there any free motions of the triangle that can not be prevented by the fingers?

Each finger applies a "wrench" to the triangle object due to its contact with the triangle. Since we are assuming a frictionless contact, the finger can only apply forces to the triangle that are normal to the triangle's boundary. Hence, each finger applies a pure force in the direction of the boundary normal vector, which corresponds to a zero pitch screw.

Define a coordinate system whose origin lies at the common intersection of all of the finger forces at the center of the triangle. Choose the z-axis of this system to be normal to the plane of the triangle. Let the x-axis coincide with one of the finger contact normals. Thus, the screw coordinates for the three wrenches are:

$$\xi_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(120^o) \\ -\sin(120^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(240^o) \\ -\sin(240^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If the triangle is not immobilized, there there must exist a twist (i.e., an instantaneous motion of the triangle) that is **reciprocal** to the finger wrenches. The $\xi_R = [0\ 0\ 1\ 0\ 0\ 0]^T$ denote the zero pitch twist that corresponds to rotation of the triangle about a vertical axis passing through the origin of the reference frame (i.e., the concurrency point of the three contact normals). This twist is reciprocal to each of the finger wrenches, and therefore, in theory, the fingers **can not** stop any rotational motions of the triangle. However, your intuition would suggest that this object is in fact immobilized. Your intuition is correct. This is somewhat of a trick question. Based purely on screw theory, the object can not be immobilized by this set of contacts. However, this theory has its limitations, as we will learn at the end of next quarter.

Problem #3: Find the Denavit-Hartenberg parameters for manipulators (ii) and (iv) in Figure 3.23 of the MLS text

(ii) The choice of the stationary frame is abitrary. We will place its origin along joint axis 1, but not necessarily at the point of coincidence of joint axes 1 and 2. The tool frame origin is placed in the middle of the "U", with its x-axis collinear with the mechanical link axis, and with its z-axis parallel to joint axis 2. In this case,

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 \neq 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 \neq 0 & \alpha_2 = -\frac{\pi}{2} & d_3 \neq 0 & \theta_3 = \text{ variable} \\ a_3 \neq 0 & \alpha_3 = \frac{\pi}{2} & d_4 = 0 & \theta_3 = \text{ constant} \end{array}$$

(iv) This is the first three joints of the "Stanford" manipulator that was reviewed in class. See your class notes for details. Also, Prob. 4 below is the same manipulator.

Problem #4: Consider the simple manipulator (iii) associated with Prob.4 in Chapter 3 of the MLS text.

- **Part (a):** *Determine the Denavit-Hartenberg parameters of this manipulator.* For simplicity, let us choose the z-axis of the stationary frame to be collinear with the first joint axis. The origin of the stationary frame is located some distance below the point of intersection of the first two axes. Also, choose the tool frame origin to coincide with the intersection point of the last three axes (the "wrist"). Also, assume that the link

6 frame of the Denavit-Hartenberg approach is the tool frame.

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 \neq 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = -\frac{\pi}{2} & d_3 = \text{variable}0 & \theta_3 = 0 \text{ (constant)} \\ a_3 = 0 & \alpha_3 = \frac{\pi}{2} & d_4 = 0 & \theta_4 = \text{ variable} \\ a_4 = 0 & \alpha_3 = \frac{\pi}{2} & d_5 = 0 & \theta_5 = \text{ variable} \\ a_5 = 0 & \alpha_3 = \frac{\pi}{2} & d_6 = 0 & \theta_6 = \text{ variable} \\ a_6 = 0 & \alpha_3 = 0 & d_7 = 0 & \theta_7 = 0 \end{array}$$

- **Part (b):** *Find the forward kinematics using the Denavit-Hartenberg approach.* To find the forward kinematics using the Denavit-Hartenberg approach, one must use the formula

$$g_{ST}(\vec{\theta}) = g_{S1}(\theta_1)g_{12}(\theta_2)g_{23}(d_3)g_{34}(\theta_4)g_{45}(\theta_5)g_{56}(\theta_6)g_{6T}.$$

where each $g_{i,i+1}$ is given by:

$$\begin{bmatrix} \cos\theta_{i+1} & -\sin\theta_{i+1}0 & a_i & \\ \sin\theta_{i+1}\cos\alpha_i & \cos\theta_{i+1}\cos\alpha_i & -\sin\alpha_i & -d_{i+1}\sin\alpha_i \\ \sin\theta_{i+1}\sin\alpha_i & \cos\theta_{i+1}\sin\alpha_i & -\cos\alpha_i & d_{i+1}\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Plugging in the D-H parameters from above yields:

$$g_{ST} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 01 & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_4 & \cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} g_{6T}$$

- **Part (c:)** *Find the forward kinematics using the Product-of-Exponentials approach.* Take the configuration in the text as the reference configuration, and place the stationary frame at the same location. Then the twists of the six joint axes are:

$$\vec{\xi}_1 = \begin{bmatrix} v_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} h_1\omega_1 + \rho_1 \times \omega_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{z}_S \end{bmatrix}$$

$$\vec{\xi}_2 = \begin{bmatrix} v_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} h_2\omega_2 + \rho_2 \times \omega_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -d_1\vec{z}_S \times \vec{y}_S \\ -\vec{y}_S \end{bmatrix} = \begin{bmatrix} d_1\vec{x}_S \\ -\vec{y}_S \end{bmatrix}$$

$$\vec{\xi}_3 = \begin{bmatrix} \omega_3 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{x}_S \\ \vec{0} \end{bmatrix}$$

$$\vec{\xi}_4 = \begin{bmatrix} v_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} h_4\omega_4 + \rho_4 \times \omega_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} -(d_1\vec{z}_S + d_3\vec{x}_S) \times \vec{y}_S \\ -\vec{y}_S \end{bmatrix} = \begin{bmatrix} d_1\vec{x}_S - d_3\vec{z}_S \\ -\vec{y}_S \end{bmatrix}$$

$$\vec{\xi}_5 = \begin{bmatrix} v_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} h_5\omega_5 + \rho_5 \times \omega_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} d_3\vec{x}_S \times \vec{z}_S \\ \vec{z}_S \end{bmatrix} = \begin{bmatrix} -d_3\vec{y}_S \\ \vec{z}_S \end{bmatrix}$$
$$\vec{\xi}_6 = \begin{bmatrix} v_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} h_6\omega_6 + \rho_6 \times \omega_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} d_1\vec{z}_S \times \vec{x}_S \\ \vec{x}_S \end{bmatrix} = \begin{bmatrix} -d_1\vec{y}_S \\ \vec{x}_S \end{bmatrix}$$

where $\vec{x}_S$, $\vec{y}_S$, and $\vec{z}_S$ are the basis vectors of the stationary frame, and assume values

$$\vec{x}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{y}_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{y}_S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The forward kinematics is then:

$$g_{ST} = e^{\theta_1\hat{\xi}_1} e^{\theta_2\hat{\xi}_2} e^{d_3\hat{\xi}_3} e^{\theta_4\hat{\xi}_4} e^{\theta_5\hat{\xi}_5} e^{\theta_5\hat{\xi}_6} g_{ST}(0)$$