ME 115(a): Solution to Homework #5

Problem #1: *Consider the two screws, S_1 and S_2, shown in Figure 1(a). The screw axis of S_1 is perpendicular to the plane, P, and has zero pitch: $h_1 = 0$. The screw axis of S_2 lies in P, and S_2 some non-zero pitch, h_2. The distance between S_1 and S_2, as measured along a mutually perpendicular line, is denoted a. Describe the set of all screws whose axes lie in P and that are reciprocal to both S_1 and S_2.*

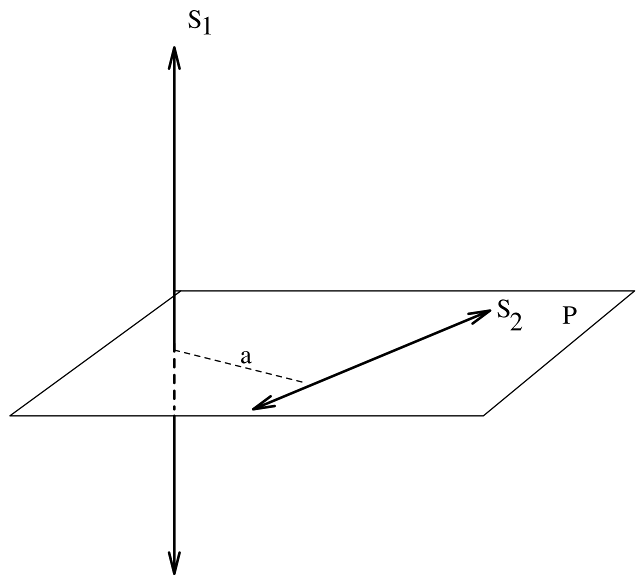

Figure 1: Two Screws.

Assign a coordinate system whose origin is located at a point C, where the screw axis of S_1 intersects the horizontal plane, P, containing S_2. Let the z-axis be collinear with the positive S_1 direction, and let the x-axis be collinear with the mutually perpendicular line between S_1 and S_2. In this coordinate system, the screw coordinates of S_1 and S_2 are:

$$\xi_1 = \begin{bmatrix} h_1\vec{\omega}_1 + \vec{\rho}_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} h_2\vec{\omega}_2 + \vec{\rho}_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ h_2 \\ a \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We require that any screw, S_R, which is reciprocal to both S_1 and S_2 also lie in the plane P. We can parametrize all screws that lie in P as follows:

$$\xi_R = \begin{bmatrix} h_R\vec{\omega}_R + \vec{\rho}_R \times \vec{\omega}_R \end{bmatrix}$$

where h_R is the pitch of the reciprocal screw while $\vec{\omega}_R$ is a unit length vector collinear with the screw axis of the reciprocal screw, $\vec{\rho}_R$ is a vector from the origin of the reference frame described above to a point on the reciprocal screw axis. By assumption, both $\vec{\omega}_R$ and $\vec{\rho}_R$

must also lie in P. We can describe any screw that lies in the plane by two scalars: d (the distance along the mutually perpendicular line between S_1 and S_R) and θ, the angle between the mutuallyperpendicular line and the x-axis of the reference coordinate system, which lies in P. In terms of these scalars:

$$\vec{\omega}_R = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad \vec{\rho}_R = d \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and hence:

$$\xi_R = \begin{bmatrix} -h_R \sin\theta \\ h_R \cos\theta \\ d \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

If S_R is reciprocal to S_1, then the reciprocal product between these two screws must be zero. Letting $\circ$ denote the reciprocal product,

$$\xi_1 \circ \xi_R = d = 0$$

This implies that the screw axis of S_R must intersect the axis of S_1. If S_R is reciprocal to S_2, then:

$$\xi_2 \circ \xi_R = (h_2 + h_R)\cos\theta = 0.$$

Hence, S_R must always intersect S_1 and either have the negative pitch of S_2, or it can have any pitch if it is orthogonal to the axis of S_2 (i.e. $\cos\theta = 0$).

Problem #2: Find the Denavit-Hartenberg parameters for manipulators (ii) and (iv) in Figure 3.23 of the MLS text

(i) The choice of the stationary frame is abitrary. For simplicity, place the origin of the stationary frame at the point where all three revolute joints intersect. Place the z-axis of the stationary frame, z_S, collinear with the first joint axis. Orient the x-axis of the stationary frame to be orthogonal to both joint axes 1 and 2 (pointing toward the right in Figure 3.23(i)). Similary, there are many choices for the tool frame. Let's assume that the tool frame is coincident with the link frame of link 3, as determined using the Denavit-Hartenberg procedure. Then, the D-H parameters are:

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = 0 & \theta_3 = \text{ variable} \\ a_3 = 0 & \alpha_3 = 0 & d_4 = 0 & \theta_4 = \text{ constant} = 0 \end{array}$$

(ii) The choice of the stationary frame is abitrary. Place its origin along joint axis 1, but not necessarily at the point of coincidence of joint axes 1 and 2. The tool frame origin is placed in the middle of the "U", with its x-axis collinear with the mechanical link axis, and with its z-axis parallel to joint axis 2. In this case,

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 \neq 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 \neq 0 & \alpha_2 = -\frac{\pi}{2} & d_3 \neq 0 & \theta_3 = \text{ variable} \\ a_3 \neq 0 & \alpha_3 = \frac{\pi}{2} & d_4 = 0 & \theta_3 = \text{ constant} \end{array}$$

where the value of d_1 will be determined by the location of the stationary frame origin.

(iv) The choice of the stationary frame is abitrary. For simplicity, place the origin of the stationary frame at the point where all three joints intersect. Place the z-axis of the stationary frame, z_S, collinear with the first joint axis. Orient the x-axis of the stationary frame to be orthogonal to both joint axes 1 and 2 (pointing toward the right in Figure 3.23(i)). Similary, there are many choices for the tool frame. Let's assume that the tool frame is parallel with the link frame of link 3, (as determined using the Denavit-Hartenberg procedure), but its origin lies at the tip of the mechanism (in the "U" of Figure 3.23(iv)). Then, the D-H parameters are:

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = \text{variable0} & \theta_3 = \ 0 \\ a_3 = 0 & \alpha_3 = 0 & d_4 = \text{constant} & \theta_3 = \text{ constant} = 0 \end{array}$$

where the constant d_4 depends upon the offset between the origin of link frame 3 and the origin of the tool frame.

Problem #3: Consider the simple manipulator (iii) associated with Prob.4 in Chapter 3 of the MLS text.

- **Part (a):** *Determine the Denavit-Hartenberg parameters of this manipulator.*

  For simplicity, let us choose the z-axis of the stationary frame to be collinear with the first joint axis. The origin of the stationary frame is located some distance below the point of intersection of the first two axes. Also, choose the tool frame origin to coincide with the intersection point of the last three axes (the "wrist"). Also, assume that the link 6 frame of the Denavit-Hartenberg approach is the tool frame.

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 \neq 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = -\frac{\pi}{2} & d_3 = \text{variable0} & \theta_3 = 0 \text{ (constant)} \\ a_3 = 0 & \alpha_3 = \frac{\pi}{2} & d_4 = 0 & \theta_4 = \text{ variable} \\ a_4 = 0 & \alpha_3 = \frac{\pi}{2} & d_5 = 0 & \theta_5 = \text{ variable} \\ a_5 = 0 & \alpha_3 = \frac{\pi}{2} & d_6 = 0 & \theta_6 = \text{ variable} \\ a_6 = 0 & \alpha_3 = 0 & d_7 = 0 & \theta_7 = 0 \end{array}$$

- **Part (b):** *Find the forward kinematics using the Denavit-Hartenberg approach.* To find the forward kinematics using the Denavit-Hartenberg approach, one must use the formula

$$g_{ST}(\vec{\theta}) = g_{S1}(\theta_1)g_{12}(\theta_2)g_{23}(d_3)g_{34}(\theta_4)g_{45}(\theta_5)g_{56}(\theta_6)g_{6T}.$$

where each $g_{i,i+1}$ is given by:

$$\begin{bmatrix} \cos\theta_{i+1} & -\sin\theta_{i+1} 0 & a_i & \\ \sin\theta_{i+1}\cos\alpha_i & \cos\theta_{i+1}\cos\alpha_i & -\sin\alpha_i & -d_{i+1}\sin\alpha_i \\ \sin\theta_{i+1}\sin\alpha_i & \cos\theta_{i+1}\sin\alpha_i & -\cos\alpha_i & d_{i+1}\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Plugging in the D-H parameters from above yields:

$$g_{ST} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 01 & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_4 & \cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} g_{6T}$$

- **Part (c:)** *Find the forward kinematics using the Product-of-Exponentials approach.* Take the configuration in the text as the reference configuration, and place the stationary frame at the same location. Then the twists of the six joint axes are:

$$\vec{\xi}_1 = \begin{bmatrix} v_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} h_1\omega_1 + \rho_1 \times \omega_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{z}_S \end{bmatrix}$$

$$\vec{\xi}_2 = \begin{bmatrix} v_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} h_2\omega_2 + \rho_2 \times \omega_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -d_1\vec{z}_S \times \vec{y}_S \\ -\vec{y}_S \end{bmatrix} = \begin{bmatrix} d_1\vec{x}_S \\ -\vec{y}_S \end{bmatrix}$$

$$\vec{\xi}_3 = \begin{bmatrix} \omega_3 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{x}_S \\ \vec{0} \end{bmatrix}$$

$$\vec{\xi}_4 = \begin{bmatrix} v_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} h_4\omega_4 + \rho_4 \times \omega_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} -(d_1\vec{z}_S + d_3\vec{x}_S) \times \vec{y}_S \\ -\vec{y}_S \end{bmatrix} = \begin{bmatrix} d_1\vec{x}_S - d_3\vec{z}_S \\ -\vec{y}_S \end{bmatrix}$$

$$\vec{\xi}_5 = \begin{bmatrix} v_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} h_5\omega_5 + \rho_5 \times \omega_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} d_3\vec{x}_S \times \vec{z}_S \\ \vec{z}_S \end{bmatrix} = \begin{bmatrix} -d_3\vec{y}_S \\ \vec{z}_S \end{bmatrix}$$

$$\vec{\xi}_6 = \begin{bmatrix} v_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} h_6\omega_6 + \rho_6 \times \omega_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} d_1\vec{z}_S \times \vec{x}_S \\ \vec{x}_S \end{bmatrix} = \begin{bmatrix} -d_1\vec{y}_S \\ \vec{x}_S \end{bmatrix}$$

where $\vec{x}_S$, $\vec{y}_S$, and $\vec{z}_S$ are the basis vectors of the stationary frame, and assume values

$$\vec{x}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{y}_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{y}_S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The forward kinematics is then:

$$g_{ST} = e^{\theta_1\hat{\xi}_1} e^{\theta_2\hat{\xi}_2} e^{d_3\hat{\xi}_3} e^{\theta_4\hat{\xi}_4} e^{\theta_5\hat{\xi}_5} e^{\theta_5\hat{\xi}_6} g_{ST}(0)$$