ME 115: Solution to Homework #5

Problem #1 (Problem #3(a) from Ch.3 of MLS, manipulator (iii)).

I'll find the forward kinematics using the Product of Exponentials approach.

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g_{st}(0)$$

where

$$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}$$

and

$$\hat{\xi}_i = \begin{bmatrix} \hat{\omega}_i & v_i \\ 0 & 0 \end{bmatrix}$$

Using the configuration in the book as the reference configuration, and placing the stationary frame at the base of the manipulator gives,

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad \hat{\xi}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\omega_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad q_2 = \begin{bmatrix} 0 \\ 0 \\ l_1 \end{bmatrix}; \quad \xi_2 = \begin{bmatrix} -l_1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \hat{\xi}_2 = \begin{bmatrix} 0 & 0 & 1 & -l_1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\omega_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad q_3 = \begin{bmatrix} l_2 \\ 0 \\ l_1 \end{bmatrix}; \quad \xi_3 = \begin{bmatrix} -l_1 \\ 0 \\ l_2 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \hat{\xi}_3 = \begin{bmatrix} 0 & 0 & 1 & -l_1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & l_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The location of the tool frame in the reference configuration is given by

$$g_{st}(0) = \begin{bmatrix} 1 & 0 & 0 & l_2 + l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, a direct calculation of the matrix exponentials gives the forward kinematics to be

$$g_{st}(\theta) = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2+\theta_3) & -\sin(\theta_1) & \cos(\theta_1)\sin(\theta_2+\theta_3) & \cos(\theta_1)(l_2\cos(\theta_2) + l_3\cos(\theta_2+\theta_3)) \\ \sin(\theta_1)\cos(\theta_2+\theta_3) & \cos(\theta_1) & \sin(\theta_1)\sin(\theta_2+\theta_3) & \sin(\theta_1)(l_2\cos(\theta_2) + l_3\cos(\theta_2+\theta_3)) \\ -\sin(\theta_2+\theta_3) & 0 & \cos(\theta_2+\theta_3) & l_1 - l_2\sin(\theta_2) - l_3\sin(\theta_2+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem #2

Assign a coordinate system whose origin is located at a point C, where the screw axis of S_1 intersects the horizontal plane, P. Let the z-axis be collinear with the positive S_1 direction, and let the x-axis be collinear with the mutually perpendicular line between S_1 and S_2. In this coordinate system, the screw coordinates of S_1 and S_2 are (recalling that $h_2 \neq h_1 = 0$ and that the screw axes are separated by a distance a):

$$\xi_1 = \begin{bmatrix} h_1\vec{\omega}_1 + \vec{\rho}_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} h_2\vec{\omega}_2 + \vec{\rho}_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ h_2 \\ a \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We require that any screw, S_R, which is reciprocal to both S_1 and S_2 also lie in the plane P.

We can parametrize all finite pitch screws that lie in P as follows:

$$\xi_R = \begin{bmatrix} h_R\vec{\omega}_R + \vec{\rho}_R \times \vec{\omega}_R \\ \vec{\omega}_R \end{bmatrix}$$

where h_R is the pitch of the reciprocal screw while $\vec{\omega}_R$ is a unit length vector collinear with the screw axis of the reciprocal screw, $\vec{\rho}_R$ is a vector from the origin of the reference frame described above to a point on the reciprocal screw axis. By assumption, both $\vec{\omega}_R$ and $\vec{\rho}_R$ must also lie in P. We can describe any screw that lies in the plane by two scalars: d (the distance along the mutually perpendicular line between S_1 and S_R) and θ, the angle between the mutually perpendicular line and the x-axis of the reference coordinate system that was defined above. In terms of these scalars:

$$\vec{\omega}_R = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad \vec{\rho}_R = d\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and hence:

$$\xi_R = \begin{bmatrix} -h_R \sin\theta \\ h_R \cos\theta \\ d \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

If S_R is reciprocal to S_1, then the reciprocal product between these two screws must be zero. Letting $\circ$ denote the reciprocal product,

$$\xi_1 \circ \xi_R = d = 0$$

This implies that the screw axis of S_R must intersect the axis of S_1. If S_R is reciprocal to S_2, then:

$$(h_2 + h_R)\cos\theta = 0.$$

Thus, S_R is reciprocal to S_2 if either $\cos\theta = 0$ (which implies $\theta = \pm\pi/2$), or if $h_R = -h_2$. Hence, a finite pitch screw is reciprocal if its axis lies along the mutual perpendicular line between S_1 and S_2, or if it intersects S_1 and has pitch $-h_2$.

Problem #3: (20 points)

Each finger applies a "wrench" to the disk object due to its contact with the disk. Since we are assuming a frictionless contact, the finger can only apply forces to the disk that are normal to the disk's boundary. Hence, each finger applies a pure force in the direction of the boundary normal vector, which corresponds to a zero pitch screw.

Define a coordinate system whose origin lies at the common intersection of all of the finger forces at the center of the disk. Choose the z-axis of this system to be normal to the plane of the disk. Let the x-axis coincide with one of the finger contact normals. Thus, the screw coordinates for the three wrenches are:

$$\xi_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(120^o) \\ -\sin(120^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(240^o) \\ -\sin(240^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If the disk is not immobilized, there there must exist a twist (i.e., an instantaneous motion of the disk) that is **reciprocal** to the finger wrenches. The $\xi_R = [0\ 0\ 0\ 0\ 0\ 1]^T$ denote the zero pitch twist that corresponds to rotation of the disk about a vertical axis passing through the origin of the reference frame (i.e., the concurrency point of the three contact normals). This twist is reciprocal to each of the finger wrenches, and therefore the fingers **can not** stop any rotational motions of the disk. Hence, the object is not immobilized.

Problem #4: (Related to Problem #4 in Chap. 3 of MLS).

Part (a): Since many of the joint axes intersect, there is not one unique set of parameters. One possible list of parameters is as follows:

$$\begin{array}{llll}
\alpha_0 = 0 & a_0 = 0 & d_1 \neq 0 & \theta_1 = \text{variable} \\
\alpha_1 = -90^o & a_1 = 0 & d_2 = 0 & \theta_2 = \text{variable} \\
\alpha_2 = -90^o & a_2 = 0 & d_3 = \text{variable} & \theta_3 = 0 \\
\alpha_3 = +90^o & a_3 = 0 & d_4 = 0 & \theta_4 = \text{variable} \\
\alpha_4 = +90^o & a_4 = 0 & d_5 = 0 & \theta_5 = \text{variable} \\
\alpha_5 = +90^o & a_5 = 0 & d_6 \neq 0 & \theta_6 = \text{variable}
\end{array}$$

Part (b): To find the forward kinematics using these parameters, one must use the formula

$$g_{st}(\theta) = g_{s1}(\theta_1)...g_{56}(\theta_6)g_{6t}$$

Where each g is given by

$$g_{i,i+1} = \begin{bmatrix} \cos\theta_{i+1} & -\sin\theta_{i+1} & 0 & a_i \\ \sin\theta_{i+1}\cos\alpha_i & \cos\theta_{i+1}\cos\alpha_i & -\sin\alpha_i & -d_{i+1}\sin\alpha_i \\ \sin\theta_{i+1}\sin\alpha_i & \cos\theta_{i+1}\sin\alpha_i & \cos\alpha_i & d_{i+1}\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Plugging in the parameters from part (a):

$$g_{st} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$
$$\begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_4 & \cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ 0 & 0 & -1 & -d_6 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot g_{6t}$$

Part (b): We find the forward kinematics using the product of exponentials approach much like we did in problem 1. Again, take the configuration in the diagram to be the reference configuration, and place the stationary frame at the base of the manipulator. Then the revolute joints are defined by the following:

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \omega_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \omega_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \omega_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \omega_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$
$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad q_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix}; \quad q_4 = \begin{bmatrix} l_3 \\ 0 \\ l_0 \end{bmatrix}; \quad q_5 = \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix}; \quad q_6 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix};$$

and

$$\xi_i = \begin{bmatrix} q_i \times \omega_i \\ \omega_i \end{bmatrix}$$

And the one prismatic joint is defined by

$$\xi_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the product of exponentials formula gives the following forward kinematics:

$$g_{st}(\theta) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 & -l_0\sin\theta_2 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & l_0(1-\cos\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$
$$\begin{bmatrix} \cos\theta_4 & 0 & \sin\theta_4 & l_3(1-\cos\theta_4) - l_0\sin\theta_4 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_4 & 0 & \cos\theta_4 & l_0(1-\cos\theta_4) + l_3\sin\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & l_3(1-\cos\theta_5 \\ \sin\theta_5 & \cos\theta_5 & 0 & -l_3\sin\theta_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_6 & -\sin\theta_6 & l_0\sin\theta_6 \\ 0 & \sin\theta_6 & \cos\theta_6 & l_0(1-\cos\theta_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot g_{st}(0)$$