ME 115: Solution to Homework #5

Problem #1: (Problem #3(b,iv) in Chap. 3 of MLS)

Manipulator (iv): The forward kinematics of this manipulator are:

$$g_{ST} = g_{S,1}g_{1,2}g_{2,3}g_{3,T}$$

where the Denavit-Hartenberg parameters that go into the matrices $g_{i,i+1}$ were given in the solution to Homework #5. To simplify things, we choose the tool frame to be coincident with the link frame of the third link (i.e., we choose $d_4 = 0$). In this case:

$$\begin{aligned} g_{ST} &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2 \\ s_1c_2 & c_1 & s_1s_2 & d_3s_1s_2 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ST} & \vec{p}_{ST} \\ \vec{0}^T & 1 \end{bmatrix} \end{aligned}$$

Inverse Kinematics: Since d_3 is variable, this manipulator is capable of reaching any desired point $\vec{p}_D = [x_D\ y_D\ z_D]^T$ within the workspace dictated by the mechanical limits of the joints. Let $\vec{p}_{ST}$ denote the portion of the forward kineatics that describes the position of the tool frame origin with respect to the stationary frame origin. Hence, we can find the inverse kinematics by equating terms of $\vec{p}_{ST}$ with $\vec{p}_D$. First notice that:

$$||\vec{p}_{ST}||^2 = d_3^2 = (x_D^2 + y_D^2 + z_D^2)$$

Hence,

$$d_3 = \pm(x_D^2 + y_D^2 + z_D^2)^{1/2} \tag{1}$$

We will denote the two possible solutions by d_3^+ and d_3^-. Next notice that $d_3\cos\theta_2 = z_D$ and $d_3^2\sin^2\theta_2 = x_D^2 + y_D^2$. Hence there are two solutions in θ_2 for a given choice of d_3:

$$\begin{aligned} \theta_2^a &= \cos^{-1}[z_D/d_3] \\ \theta_2^b &= -\theta_2^a \end{aligned}$$

However, there at not four possible solutions, since two of the solutions are the same. Hence, there are two combinations of θ_2 and d_3. Finally, we can determine the value of θ_1 from the x and y components of $\vec{p}_{ST}$:

$$\begin{aligned} \cos\theta_1 &= \tfrac{x_D}{d_3\sin\theta_2} \\ \sin\theta_1 &= \tfrac{y_D}{d_3\sin\theta_2} \\ \theta_1 &= Atan2[\cos\theta_1, \sin\theta_1] \end{aligned}$$

There will be one θ_1 solution for each of the two (θ_2, d_3) pairs.

Problem #2: The goal of this problem is to find the numerical values of the inverse kinematics for Manipulator (iii) in Figure 3.23 (the "elbow manipulator") and Manipulatvor (iv).

Manipulator (iii): Recall from the class notes that the position part of the forward kinematics is:

$$\begin{aligned} x_T &= (a_2 c_2 + a_3 c_{23}) c_1 && (2) \\ y_T &= (a_2 c_2 + a_3 c_{23}) s_1 && (3) \\ z_T &= -(a_2 s_2 + a_3 s_{23}) && (4) \end{aligned}$$

where $s_j = \sin\theta_j$, $c_j = \cos\theta_j$, and $s_{ij} = \sin(\theta_i + \theta_j)$. From the class notes, we have that squaring and summing Equations (2), (3), and (4) yields

$$\cos\theta_3 = \frac{R^2 - a_2^2 - a_3^2}{2a_2 a_3} = \frac{2-1-1}{2 \cdot 1 \cdot 1} = 0$$

where $R^2 = x_T^2 + y_T^2 + z_T^2$, and (x_T, y_T, z_T) is the desired tool frame location, which is $(1,1,0)$ in this case. If $\cos\theta_3 = 0$, then there will be two solutions for θ_3: $\theta_{3a} = +90^o$ and $\theta_{3b} = -90^o$. Here we will deviate from the class notes and solve for θ_2 in a slightly different way by using Equation (4):

$$s_2(a_2 + a_3 c_3) + c_2(a_3 s_3) = -z_T$$

Substituting in the constants $a_2 = 1$, $a_3 = 1$, and $z_T = 0$ yields these two equations for θ_{3a} and θ_{3b}:

$$\begin{aligned} s_2 + c_2 &= 0 \Rightarrow \frac{2q}{1+1^2} + \frac{1-q^2}{1+q^2} = 0 && (5) \\ s_2 - c_2 &= 0 \Rightarrow \frac{2q}{1+1^2} - \frac{1-q^2}{1+q^2} = 0 && (6) \end{aligned}$$

where $q_2 = \tan(\frac{\theta_2}{2})$. From the first equation, we get that for $\theta_3 = \theta_{3a}$, there are two solutions $\tan(\frac{\theta_2}{2}) = 1 \pm \sqrt{q}$. Denote these two solutions as θ_{2aa} and θ_{2ab}: $\theta_{2aa} = 135^0$, $\theta_{2ab} = -45^o$. From the second equation, we get that for $\theta_3 = \theta_{3b}$, there are two solutions $\tan(\frac{\theta_2}{2}) = -1 \pm \sqrt{q}$. Denote these two solutions as θ_{2ba} and θ_{2bb}: $\theta_{2ba} = 45^0$, $\theta_{2bb} = -135^o$.

Again following the class notes, we have from Equations (2) and (3):

$$\begin{aligned} \cos\theta_1 &= \frac{x_T}{(a_2 c_2 + a_3 c_{23})} && (7) \\ \sin\theta_1 &= \frac{y_T}{(a_2 c_2 + a_3 c_{23})} && (8) \end{aligned}$$

Substituting each of the values of θ_2 and θ_3 into these equations yields these four solutions:

$$(-135^o, 135^o, 90^o) \qquad (9)$$

$$(45^o, -45^o, 90^o) \tag{10}$$
$$(45^o, 45^o, -90^o) \tag{11}$$
$$(-135^o, -135^o, -90^o) \tag{12}$$

Manipulator (iii): Here we follow directly from the solution to Problem 1. From Equation (1) we have

$$d_3 = \pm\sqrt{1+1} = \pm\sqrt{2}.$$

From Equation () we have

$$\cos\theta_2 = \frac{z_T}{d_3} = 0; \Rightarrow \theta_{2a} = 90^o, \theta_{2b} = -90^o.$$

Using Equation (), we can piece together all of the solutions:

$$(45^o, 90^o, \sqrt{2}) \quad (-135^o, -90^o, -\sqrt{2})$$

Problem #3: (Problem #3(c) in Chap. 3 of MLS, manipulators (i) and (iv))

Recall that the spatial Jacobian has the form:

$$J^s_{ST} = \begin{bmatrix} \xi_1 \; \xi_2^{'} \; \xi_3^{'} \end{bmatrix}.$$

where the ξ_i are the twist coordinates of the i^{th} joint axis in the manipulator's nominal configuration, and $\xi_i^{'} = Ad_{e^{\theta_1 \hat{\xi}_1} \cdots e^{\theta_{i-1}\hat{\xi}_{i-1}}} \xi_i$.

We can directly compute the twist vectors for the two revolute joints using the formula:

$$\xi_i = \begin{bmatrix} h_i \vec{\omega}_i + \vec{\rho}_i \times \vec{\omega}_i \\ \vec{\omega}_i \end{bmatrix}$$

where $\vec{\omega}_i$ is a unit vector collinear with the joint axis, h_i is the pitch of the joint axis, which is zero for the case of revolute joints, and ρ_i is a vector from the origin of the spatial reference frame (the robot's stationary frame) to a point on the i^{th} joint axis.

Manipulator (i) Spatial Jacobian: In the manipulator's nominal configuration,

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and:

$$\xi_2^{'} = Ad_{g_{S,1}} \xi_2 = \begin{bmatrix} R_{S,1} & 0 \\ 0 & R_{S,1} \end{bmatrix} \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}$$

$$\xi_3' = Ad_{g_{S,2}}\xi_3 = \begin{bmatrix} R_{S,2} & 0 \\ 0 & R_{S,2} \end{bmatrix}\xi_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_1c_2 \\ s_1c_2 \\ -s_2 \end{bmatrix}$$

where

$$R_{S,1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{S,2} = R_{S,1}R_{1,2} = R_{S,1}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ 0 & 0 & 1 \\ -\sin\theta_2 & -\cos\theta_2 & 0 \end{bmatrix} = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 \\ s_1c_2 & -s_1s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix}$$

Hence, the spatial Jacobian is:

$$J_{ST}^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s_1 & c_1c_2 \\ 0 & c_1 & s_1c_2 \\ 1 & 0 & -s_2 \end{bmatrix}$$

Manipulator (iv) Spatial Jacobian: The twist for the prismatic joint is:

$$\xi_3 = \begin{bmatrix} \vec{\omega}_3 \\ \vec{0} \end{bmatrix}$$

where $\vec{\omega}_3$ is a unit vector collinear with the axis of translation.

Since joint axes 1 and 2 intersect the stationary frame origind, these twists can be determined by inspection:

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}$$

The twist corresponding to the prismatic joint is:

$$\xi_3 = \begin{bmatrix} \cos\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2 \\ \cos\theta_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$