ME 115(b): Solution to Problem Set #6

Problem #1: Problem 14 in MLS text.

The following equations will be used for both the ellipsoid and the torus:

$$n = \frac{c_u * c_v}{|c_u * c_v|}$$

$$I_p = \begin{bmatrix} c_u^T c_u & c_u^T c_v \\ c_v^t c_u & c_v^t c_v \end{bmatrix}$$

$$II_p = \begin{bmatrix} c_u^T n_u & c_u^T n_v \\ c_v^t n_u & c_v^t n_v \end{bmatrix}$$

$$M_p M_p = I_p$$

$$K_p = M_p^{-T} II_p M_p^{-1}$$

and, if the chart is orthogonal,

$$T_p = \begin{bmatrix} \frac{c_v^T c_{uu}}{||c_u||^2 ||c_v||} & \frac{c_v^T c_{uv}}{||c_u|| ||c_v||^2} \end{bmatrix}$$

where the subscripts indicate partial derivatives.

Ellipsoid: We'll use the chart:

$$c(u,v) = \begin{bmatrix} \rho_x \cos(u)\cos(v) \\ \rho_y \cos(u)\sin(v) \\ \rho_z \sin(u) \end{bmatrix}$$

Note that this is not, in general, orthogonal. However, we can make it orthogonal by assuming that $\rho_x = \rho_y = \rho_{xy}$. Let us make this assumption, as it will greatly ease the calculations. Therefore,

$$M_p = \begin{bmatrix} \sqrt{\rho_{xy}^2 \sin^2(u) + \rho_z^2 \cos^2(v)} & 0 \\ 0 & \rho_z |\cos(u)| \end{bmatrix}$$

$$K_p = \begin{bmatrix} \frac{-\rho_{xy}\rho_z}{(\rho_{xy}^2 \sin^2(u) + \rho_z^2 \cos^2(u))^{3/2}} & 0 \\ 0 & \frac{-\rho_z}{\rho_{xy}(\rho_{xy}^2 \sin^2(u) + \rho_z^2 \cos^2(u))^{1/2}} \end{bmatrix}$$

$$T_p = \begin{bmatrix} 0 & \frac{-\tan(u)}{(\rho_{xy}^2 \sin^2(u) + \rho_z^2 \cos^2(u))^{1/2}} \end{bmatrix}$$

Torus: For the torus, we'll use the orthogonal chart:

$$c(u,v) = \begin{bmatrix} \cos(u)(a + b\cos(v)) \\ \sin(u)(a + b\cos(v)) \\ b\sin(v) \end{bmatrix}$$

It is easy to show that this chart is, in fact orthogonal.

$$M_p = \begin{bmatrix} a + b\cos(v) & 0 \\ 0 & b \end{bmatrix}$$

$$K_p = \begin{bmatrix} \frac{\cos(v)}{a+b\cos(v)} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$$

$$T_p = \begin{bmatrix} \frac{\sin(v)}{a+b\cos(v)} & 0 \end{bmatrix}$$

Problem #2:

We'll parameterize the two ellipses as:

$$p_1(s_1) = \begin{bmatrix} a_1 \cos(s_1) \\ b_1 \sin(s_1) \end{bmatrix}$$
$$p_2(s_2) = \begin{bmatrix} a_2 \cos(s_2) \\ b_2 \sin(s_2) \end{bmatrix}$$

The rolling/sliding contact equations are, as given in the handout,

$$\dot{s}_1 = M_1^{-1}(\kappa_1 + \kappa_2)^{-1}(\dot{\theta}_{12} + \kappa_2 v_t)$$
$$\dot{s}_2 = M_2^{-1}(\kappa_1 + \kappa_2)^{-1}(-\dot{\theta}_{12} + \kappa_1 v_t)$$

Where

$$M_i = \sqrt{a_i^2 \sin^2(s_i) + b_i^2 \cos^2(s_i)}$$

and,

$$\kappa_i = -M_i^{-3}(a_i b_i)$$

In the position shown, $s_1 = 0$ and $s_2 = \pi/2$. Thus,

$$\begin{array}{ll} M_1 = b_1 & M_2 = a_2 \\ \kappa_1 = \frac{-a_1}{b_1^2} & \kappa_2 = \frac{-b_2}{a_2^2} \end{array}$$

And, the relative curvature is given by $\kappa_1 + \kappa_2 = -(\frac{a_1}{b_1^2} + \frac{b_2}{a_2^2})$. This is ill defined when either $b_1 = 0$ or $a_2 = 0$, corresponding to the contact point located at the end on an infinitely thin flat plate.

Problem #3:

This problem was omitted from the homework, as the given chart was not orthogonal.