ME 115(a): Homework #3

(Due Friday Feb. 5, 2010)

Problem 1: (20 Points). Consider 2×2 complex matrices of the form:

$$M = \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} = \begin{bmatrix} (a+ib) & (c+id) \\ -(c-id) & (a-ib) \end{bmatrix}$$

where z and w are complex, and a, b, c, and d are real numbers. Further, the variables z and w are constrained such that:

$$det(M) = zz^* + ww^* = 1$$

and $z, w \in \mathbb{C}$, and $*$ denotes complex conjugation. Such matrices form a matrix group termed the "special unitary matrices" of dimension 2, $SU(2)$.

- **Part (a):** Show that matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

  form a basis for $SU(2)$. The element i is $\sqrt{-1}$. I.e., show that all elements of $SU(2)$ can be expressed as some linear combination of these basis elements. Next show that elements of $SU(2)$ are isomorphic to the unit quaternions. That is, there is a one-to-one correspondence between each element of $SU(2)$ and a unit quaternion.

- **Part (b):** Show that the special unitary representation of a rotation in terms of z-y-x Euler Angles can be computed as :

$$\begin{bmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{bmatrix}$$

  where ψ, ϕ, and γ are respectively the rotations about the z, y, and x axes.

- **Part (c):** Suppose a rotation is represented by a special unitary matrix of the form:

$$\begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix}$$

  determine the angle of rotation, ϕ, and the axis of rotation, $\hat{s}$.

Problem 2: (10 points). Can every orthogonal matrix be represented by a the exponential of a real matrix? That is, if $A \in \mathcal{O}(n)$, can A be represented by

$$A = e^C$$

for some *real* matrix C? (Hint: the determinant of e^C can be expressed as an exponential.)

Problem 3: (5 points) Problem 5(c) in Chapter 2 of MLS.

Problem 4: (10 points) Problem 6(a,d,e) in Chapter 2 of MLS.

Problem 5: (10 points) Problem 8(b,c) in Chapter 2 of MLS.

Problem 6: (10 points) Problem 9(a,b) in Chapter 2 of MLS.

Problem 7: (10 points) Given a rotation matrix $A \in SO(3)$ of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Find the z-y-x Euler Angles given the matrix entries $\{a_{ij}\}$.