ME 115(b): Problem Set #5
(Due May 30, 2014)

Problem #1: (15 points)

Consider an ellipse rolling/sliding on a plane (see Figure 1). The boundary of the ellipse can be parametrized as:

$$\begin{bmatrix} x(u) \\ y(u) \end{bmatrix} = \begin{bmatrix} a\cos(u) \\ b\sin(u) \end{bmatrix} \ . \tag{1}$$

Note that this parametrization is not an arc-length parametrizatrion. Also note that when $a = b$, the equations reduce to the case of a disc rolling on a plane.

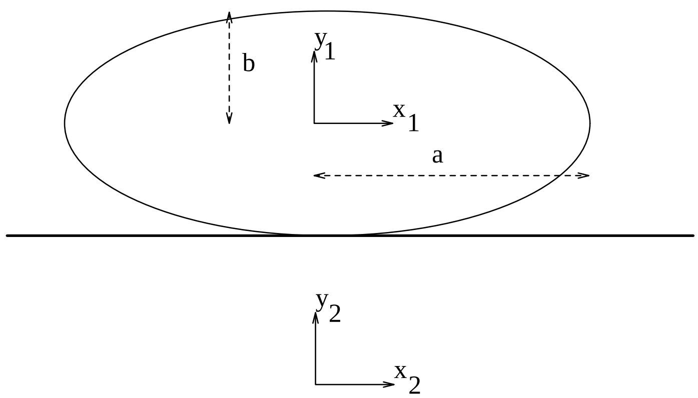

Figure 1: Planar Ellipse rolling on a surface

Part (a): Derive the contact equations for the roll/sliding ellipse on the plane.

Part (b): Derive the contact equations for an ellipse rolling/sliding on a circle of radius R. Under what conditions does the relative curvature become ill defined?

Problem #1 (30 points)

This problem introduces you to the concepts of the *involute* and the *evolute* of a planar curve. Let $\alpha\colon I = (a, b) \rightarrow \mathbb{R}^2$ be a regular parametrized plane curve where the curve paramters t is not necessarily an arc-length parametrization. The **involute** curve, $\gamma(t) = (\gamma_x(t), \gamma_y(t))$, of the curve $\alpha(t) = (x(t), y(t))$ is given by:

$$\begin{aligned} \gamma_x(t) &= x(t) - \frac{x'}{\sqrt{(x')^2 + (y')^2}} \int_a^t \sqrt{(x')^2 + (y')^2}\ dt && (2) \\ \gamma_y(t) &= y(t) - \frac{y'}{\sqrt{(x')^2 + (y')^2}} \int_a^t \sqrt{(x')^2 + (y')^2}\ dt\ . && (3) \\ & && (4) \end{aligned}$$

Practically, one can interpret the involute of a curve $\alpha(t)$ is the new curved obtained by "unwrapping" a string from the boundary of $\alpha(t)$ while keeping the string taut. The **evolute** of the curve $\alpha(t)$, denoted $\beta(t)$ is defined by:

$$\beta(t) = \alpha(t) + \frac{1}{\kappa(t)}\vec{n}(t) \ . \tag{5}$$

$\kappa(t)$ is the curvature at t, while $\vec{n}$ is the unit normal vector at t. One can think of the evolute curve as the curve which defines the loci of the centers of curvature of α.

(a) Show that the tangent at curve parameter t of the evolute of any arbitrary curve $\alpha(t)$ is the normal to α at t.

(b) What is the evolute of a circle?

Involutes and evolutes are particulary important in gear theory, as the vast majory of gear teeth profiles are the involute curves of a circle. A parametrized formula for the involute curve of a circle is:

$$x(t) = R(\cos(t) + t\sin(t)) \tag{6}$$
$$x(t) = R(\sin(t) - t\cos(t)) \tag{7}$$

where R is the radius of the circle which defines the involute.

(c) Show that the evolute of the gear tooth profile is a circle of radius R.

(d) Using the class handout on the planar contact equations, find an expression for the curvature, $\kappa(t)$, and length of the tangent vector, $M(t) = ||d\gamma(t)/dt||$, of the evolute as a function of t.

(e) Consider two gears contacting each other. Assume each gear has exactly the same *base circle* (that is the circle of radius R in (6). Also assume that the gears contact each other at the same point along their respect involute profiles: $t_1 = t_2$, where t_i is the location of the contact point on the i^{th} body in curve parameter coordinates. Show that if $\dot{s}_1 = -\dot{s}_2$, then the two gear bodies move with respect to each other using pure rollling–i.e., there is no slide of the gears.

(f) Assuming the same contact conditions as part (e), and assuming that their is no relative sliding of the two bodies, compute $\dot{s}_i$ as a function of the gear's rotation rate and the contact paramter t.

Extra Credit: (5 points) A planar ellipse can be easily parametrized as

$$\alpha(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix} \tag{8}$$

where a and b are the major and minor principal dimensions of the ellipse. Show that the evolute of the planar ellipse is an **astroid**,

$$\beta(t) \;=\; \begin{bmatrix} \frac{(a^2-b^2)\cos^3(t)}{a} & \frac{(b^2-a^2)\sin^3(t)}{b} \end{bmatrix} \tag{9}$$

and plot the evolute and the ellipse.

Problem #3: Problem 15(b), chapter 5 of MLS.

For an ellipse with principle dimensions 2A, 2B, 2C (with $A > B > C$), note that the surface can be covered by the 8 *orthogonal* coordinate patches:

$$f(u,v) = \begin{bmatrix} \pm A\sqrt{\frac{(A-u)(A-v)}{(A-B)(A-C)}} \\ \pm B\sqrt{\frac{(B-u)(B-v)}{(B-A)(B-C)}} \\ \pm C\sqrt{\frac{(C-u)(C-v)}{(C-A)(C-B)}} \end{bmatrix} \tag{10}$$

where $u \in [C, B]$, $v \in [B, A]$. It can be shown that these coordinates are indeed orthogonal.