ME 115(b): Problem Set #5
(Due May 18, 2016)

Problem #1 (20 points): Problem 10 (b) in Chapter 5 of MLS. Assume that the disk-like finger tips make a point contact with friction contact with rectangular grasped object.

Problem #2 (30 points)

This problem introduces you to the concepts of the *involute* and the *evolute* of a planar curve. Let $\alpha\colon I = (a, b) \to \mathbb{R}^2$ be a regular parametrized plane curve where the curve parameters t is not necessarily an arc-length parametrization. The **involute** curve, $\gamma(t) = (\gamma_x(t), \gamma_y(t))$, of the curve $\alpha(t) = (x(t), y(t))$ is given by:

$$\gamma_x(t) = x(t) - \frac{x'}{\sqrt{(x')^2 + (y')^2}} \int_a^t \sqrt{(x')^2 + (y')^2}\, dt \tag{1}$$

$$\gamma_y(t) = y(t) - \frac{y'}{\sqrt{(x')^2 + (y')^2}} \int_a^t \sqrt{(x')^2 + (y')^2}\, dt\ . \tag{2}$$

Practically, one can interpret the involute of a curve $\alpha(t)$ is the new curved obtained by "unwrapping" a string from the boundary of $\alpha(t)$ while keeping the string taut. The **evolute** of the curve $\alpha(t)$, denoted $\beta(t)$ is defined by:

$$\beta(t) = \alpha(t) + \frac{1}{\kappa(t)}\vec{n}(t)\ . \tag{3}$$

$\kappa(t)$ is the curvature at t, while $\vec{n}$ is the unit normal vector at t. One can think of the evolute curve as the curve which defines the loci of the centers of curvature of α.

(a) Show that the tangent at curve parameter t of the evolute of any arbitrary curve $\alpha(t)$ is the normal to α at t.

(b) What is the evolute of a circle?

Involutes and evolutes are particulary important in gear theory. The vast majory of gear teeth profiles are the involute curves of a circle, as this is one of the tooth profiles that satisfies the *gearing principle*. A parametrized formula for the involute curve of a circle is:

$$x(t) = R(\cos(t) + t\sin(t)) \tag{4}$$

$$x(t) = R(\sin(t) - t\cos(t)) \tag{5}$$

where R is the radius of the circle which defines the involute.

(c) Show that the evolute of the gear tooth profile is a circle of radius R.

(d) Find an expression for the curvature, $\kappa(t)$, and length of the tangent vector, $M(t) = ||d\gamma(t)/dt||$, of the evolute as a function of t.

Extra Credit: (5 points) A planar ellipse can be easily parametrized as

$$\alpha(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix} \tag{6}$$

where a and b are the major and minor principal dimensions of the ellipse. Show that the evolute of the planar ellipse is an **astroid**,

$$\beta(t) \;=\; \begin{bmatrix} \frac{(a^2-b^2)\cos^3(t)}{a} & \frac{(b^2-a^2)\sin^3(t)}{b} \end{bmatrix} \tag{7}$$

and plot the evolute and the ellipse.