ME 115(a): Solution to Homework #1

Problem 1:

Part b: Recall that the location of the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position ${}^{B}\vec{p}$ as seen by an observer in the fixed B frame. After displacement, the observer in the body fixed C frame also sees the pole in his/her coordinates at point ${}^{B}\vec{p}$. However, the moving body has displaced relative to the fixed observer by amount $D_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by a coodinate transform. Since the pole is at the same location in both the fixed and moving frames, it must be true that:

$$^{B}\vec{p} = \vec{d}_{12} + R_{12} \;\; {}^{B}\vec{p}.$$

This equation can be solved to find the pole location:

$$^{B}\vec{p} = (I - R_{12})^{-1}\vec{d}_{12}$$

(Of course, you need to show the fact that $(I - R_{12})$ is invertible. It will always be invertible, except when $R_{12} = I$. In this case, the motion is a pure translation, and the pole is the "pole at infinity.")

Part a: From the problem statement we know the relationship between coordinate frames A and B. Thus:

$$^{A}\vec{p} = \vec{d}_{01} + R_{01}{}^{B}\vec{p}.$$

The problem statement asks for ${}^{A}\vec{p}$ as a function of R_{01}, R_{12}, $\vec{d}_{01}$, and $\vec{d}_{12}$. This can be done by substituting in the statement for ${}^{B}\vec{p}$ that was found above.

$$^{A}\vec{p} = \vec{d}_{01} + R_{01}(I - R_{12})^{-1}\vec{d}_{12}$$

Part c: Since reference frame B is fixed to the rigid body,

$$^{C}\vec{p} =^{B} \vec{p}$$

Problem 2: To find the pole of the displacement: $D_1 = (x, y, \theta) = (1.0, 2.0, 30.0^o)$ and $D_2 = (x, y, \theta) = (2.0, 2.0, 45.0^o)$, substitute into the above results:

$$^{B}\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \cos(45^o) & -\sin(45^o) \\ \sin(45^o) & \cos(45^o) \end{bmatrix} \right)^{-1} \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix} = \begin{bmatrix} -1.4142 \\ 3.4142 \end{bmatrix}$$

$$^{A}\vec{p} = \vec{d}_{01} + R_{01}{}^{B}\vec{p} = \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix} + \begin{bmatrix} \cos(30^o) & -\sin(30^o) \\ \sin(30^o) & \cos(30^o) \end{bmatrix} \begin{bmatrix} -1.4142 \\ 3.4142 \end{bmatrix} = \begin{bmatrix} -1.9319 \\ 4.2497 \end{bmatrix}$$

As we showed in Problem 1, ${}^{B}\vec{p} ={}^{C} \vec{p}$.

Problem 3: To show that a transformation is a pure rotation when viewed in a reference frame at the pole, select a new reference frame, denoted by D, whose basis vectors are parallel to Frame B and whose origin lies at the pole of the displacement. Let $\vec{p}$ denote the location of the pole, as seen by an observer in Frame B. The location of Frame B relative to Frame D is a pure translation of amount $-^1\vec{p}$, and therefore, $D_{DB} = (-\vec{p}, I)$. The displacement of the body from the first position to the second position, as now observed in Frame D, is obtained by a similarity transform $D_{DB}D_{12}D_{DB}^{-1}$:

$$\begin{aligned}
D_{DB}D_{12}D_{DB}^{-1} &= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(-\vec{p}, I)^{-1} && (1)\\
&= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(+\vec{p}, I) && (2)\\
&= (-\vec{p}, I)((\vec{d}_{12} + R_{12}\vec{p}), R_{12}) && (3)\\
&= ((\vec{d}_{12} + (R_{12} - I)\vec{p}), R_{12}) && (4)
\end{aligned}$$

Hence, if $\vec{p} = -(R_{12} - I)^{-1}\vec{d}_{12} = (I - R_{12})^{-1}\vec{d}_{12}$, then $D_{DB}D_{12}D_{DB}^{-1} = (\vec{0}, R_{12})$. I.e., as viewed in reference Frame D, the displacement is a pure rotation by amount R_{12}.

Problem 4: (problem 3(c) in chapter 2 of the MLS text). Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 & . \end{bmatrix}$$

Expanding $det(R)$ using cofactors, one finds:

$$\begin{aligned}
det(R) &= r_{11}(r_{22}r_{33} - r_{32}r_{23}) + r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{12}r_{23} - r_{22}r_{13})\\
&= \vec{r}_1^{\,T} \cdot (\vec{r}_2 \times \vec{r}_3)
\end{aligned}$$

Problem 5:

Part (a):

There are many ways that one can prove that reflections preserve length. Here is one approach (see Figure 1).

Select any two non-identical points, A and B, in a rigid body. After reflection, those points become $A^{'}$ and $B^{'}$. Form the right triangle ABD, where the line BD is chosen to be perpendicular to the line $AA^{'}$. Similary, in the reflected body, form the right triangle $A^{'}B^{'}D^{'}$. Simple geometric arguments show that since the distance $|BD|$ and $|B^{'}D^{'}|$ are equal, and the distances $|AD|$ and $|A^{'}D^{'}|$ are equal, then $|AB| = |A^{'}B^{'}|$. Hence, the distance between A and B is preserved under reflection. Since A and B were chosen randomly, the result will

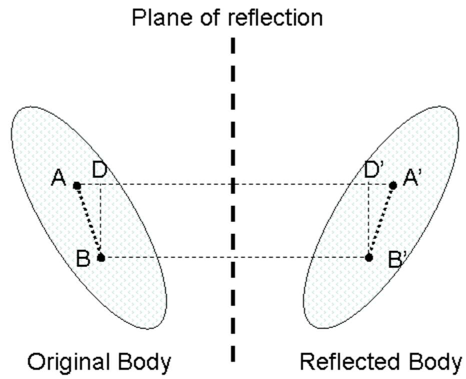

Figure 1: Diagram of reflection process

hold for any non-identical pair of points in the body. Thus, distance is always preserved under reflection.

Part (b): Generally, physically meaningful planar displacements are not equivalent to a single reflection. To see this, define three points (A, B, C) in the body of Figure 1. Because the body is rigid, one can think of points (A, B, C) as forming a rigid triangle. Consider the triangle formed from the reflected points $(A^{'}, B^{'}, C^{'})$. Note that it is impossible physically translate (A, B, C) to $(A^{'}, B^{'}, C^{'})$. Finally, note that any rigid body planar displacement can generally be realized as the result of two sequential reflections.

Problem 6: You were to "prove" that a body undergoing spherical motion has three degrees of freedom.

A body undergoing spherical motion has one fixed point. Let the body consist of N particles. Let P_1 denote the particle lying at the fixed point. A point in 3-dimensional Euclidean space normally requires 3 independent variables to fix its location. However, since P_1 does not move, it actually has 0 degrees-of-freedom (DOF). Now consider a particle P_2 in the body. Particle P_2 has 3 DOF as a particle. However, it is constrained to lie a fixed distance, d_{12} from particle P_1 due to the fact that P_1 and P_2 are part of the same rigid body. The fixed distance relationship imposes one constraint on P_2. Next consider a point P_3, which lie a fixed distance from P_1 and P_2. Therefore, there are two constraints on its location. Now, consider a particle P_4. Since its must lie a fixed distance from P_1, P_2, and P_3, there are three constraints on its motion. Particles P_5, $\ldots$, P_N similarly have 3 constraints.

The total number of degrees of freedom of the N particles are: $3(N-1)+0 = 3N-3$. The total number of constraints on these particles are: $1 + 2 + 3(N-3) = 3N-6$. Hence, the total net DOF of a body is the number of freedoms of the particles minus the number of constraints that bind them into a rigid body: $(3N-3) - (3N-6) = 3$.