ME 115(a): Solution to Homework #6

Problem #1:

Part (a): Using the Denavit-Hartenberg approach, the forward kinematics of this manipulator are:

$$g_{ST} = g_{S,1}g_{1,2}g_{2,3}g_{3,T}$$

where the matrices $g_{i,i+1}$ define the transformations between adjacent link frames. To simplify things, we choose the tool frame to be parallel with the link frame of the third link, but displaced a distance d_4 along the link frame z-axis. In this case:

$$\begin{aligned} g_{ST} &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & (d_3+d_4)c_1s_2 \\ s_1c_2 & c_1 & s_1s_2 & (d_3+d_4)s_1s_2 \\ -s_2 & 0 & c_2 & (d_3+d_4)c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ST} & \vec{p}_{ST} \\ \vec{0}^T & 1 \end{bmatrix} \end{aligned}$$

Inverse Kinematics: Since d_3 is variable, this manipulator is capable of reaching any desired point $\vec{p}_D = [x_D\ y_D\ z_D]^T$ within the workspace dictated by the mechanical limits of the joints. Let $\vec{p}_{ST}$ denote the portion of the forward kineatics that describes the position of the tool frame origin with respect to the stationary frame origin. Hence, we can find the inverse kinematics by equating terms of $\vec{p}_{ST}$ with $\vec{p}_D$. First notice that:

$$||\vec{p}_{ST}||^2 = (d_3+d_4)^2 = (x_D^2 + y_D^2 + z_D^2)$$

Hence,

$$d_3 = -d_4 \pm (x_D^2 + y_D^2 + z_D^2)^{1/2} \tag{1}$$

We will denote the two possible solutions by d_3^+ and d_3^-. Next notice that $(d_3+d_4)\cos\theta_2 = z_D$ and $(d_3+d_4)^2\sin^2\theta_2 = x_D^2 + y_D^2$. Hence there are two solutions in θ_2 for a given choice of d_3:

$$\begin{aligned} \theta_2^a &= \cos^{-1}[z_D/(d_3+d_4)] \\ \theta_2^b &= -\theta_2^a \end{aligned}$$

However, there at not four possible solutions, since two of the solutions are the same. Hence, there are two combinations of θ_2 and d_3. Finally, we can determine the value of θ_1 from the x and y components of $\vec{p}_{ST}$:

$$\begin{aligned} \cos\theta_1 &= \frac{x_D}{(d_3+d_4)\sin\theta_2} \\ \sin\theta_1 &= \frac{y_D}{(d_3+d_4)\sin\theta_2} \\ \theta_1 &= Atan2[\cos\theta_1, \sin\theta_1] \end{aligned}$$

There will be one θ_1 solution for each of the two (θ_2, d_3) pairs.
Part (b):

First, find the D-H parameters.

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = \theta_3 & \theta_3 = 0 \\ a_3 = 0 & \alpha_3 = \frac{\pi}{2} & d_4 = 0 & \theta_4 = \text{ variable} \\ a_4 = 0 & \alpha_4 = \frac{\pi}{2} & d_5 = 0 & \theta_5 = \text{ variable} \\ a_5 = 0 & \alpha_5 = \frac{\pi}{2} & d_6 = l_6 & \theta_5 = \text{ variable} \end{array}$$

To find the forward kinematics using these parameters, one must use the formula

$$g_{ST}(\theta) = g_{ST}(\theta_1)...g_{56}(\theta_6)g_{6T} \tag{2}$$

where each g is given by:

$$g_{i,i+1} = \begin{bmatrix} cos\theta_{i+1} & -sin\theta_{i+1} & 0 & \alpha_i \\ sin\theta_{i+1}cos\alpha_i & cos\theta_{i+1}cos\alpha_i & -sin\alpha_i & -d_{i+1}sin\alpha_i \\ sin\theta_{i+1}sin\alpha_i & cos\theta_{i+1}sin\alpha_i & cos\alpha_i & d_{i+1}cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$g_{ST} = \begin{bmatrix} cos\theta_1 & -sin\theta_1 & 0 & 0 \\ sin\theta_1 & cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_2 & -sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -sin\theta_2 & -cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_4 & -sin\theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ sin\theta_4 & cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} cos\theta_5 & -sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ sin\theta_5 & cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_6 & -sin\theta_6 & 0 & 0 \\ 0 & 0 & -1 & -l_6 \\ sin\theta_6 & cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}] [$$

θ_1, θ_2, and θ_3 will be the same as in part (a). We can also easily calculate R_{S3} (which we got in part (a).)

$$R_{S3} = R_{S1}R_{12}R_{23} = \begin{bmatrix} cos\theta_1cos\theta_2 & -sin\theta_1 & cos\theta_1sin\theta_2 \\ sin\theta_1cos\theta_2 & cos\theta_1 & sin\theta_1sin\theta_2 \\ -sin\theta_2 & 0 & cos\theta_2 \end{bmatrix}$$

Similarly,

$$R_{3T} = R_{34}R_{45}R_{56} = \begin{bmatrix} cos\theta_4cos\theta_5cos\theta_6 + sin\theta_4sin\theta_6 & -cos\theta_4cos\theta_5sin\theta_6 + sin\theta_4cos\theta_6 & cos\theta_4sin\theta_5 \\ -sin\theta_5cos\theta_6 & sin\theta_5sin\theta_6 & cos\theta_5 \\ sin\theta_4cos\theta_5cos\theta_6 - cos\theta_4sin\theta_6 & -sin\theta_4cos\theta_5sin\theta_6 - cos\theta_4cos\theta_6 & sin\theta_4sin\theta_5 \end{bmatrix} \tag{3}$$

We're interested in the inverse kinematics, so we want to know what combinations of angles will give R_D (R desired) and d_D (d desired). The inverse kinematics for position depend only on θ_1, θ_2 and θ_3 and thus are the same as in part (a). (This is because the "regional" part of this manipulator is exactly the same as in part (a).)

$$g_{S6} = \begin{bmatrix} R_{S6} & d_{S6} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_D & d_D \\ 0 & 1 \end{bmatrix}$$

We know that R_{S6}=R_{S1} R_{12} R_{23} R_{34} R_{45} R_{56} and thus $R_{S6} = R_{S3}$ R_{46}.

R_{S3} is invertible,so R_{S3}^{-1} =R_{S3}^T

$$R_{46} = R_{13}^T R_D. \tag{4}$$

$$R_D = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$R_{46} = \begin{bmatrix} cos\theta_1 cos\theta_2 & -sin\theta_1 & cos\theta_1 sin\theta_2 \\ sin\theta_1 cos\theta_2 & cos\theta_1 & sin\theta_1 sin\theta_2 \\ -l_1 sin\theta_2 & 0 & l_1 cos\theta_2 \end{bmatrix}^{-1} \begin{bmatrix} D \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}cos\theta_1 cos\theta_2 - sin\theta_1 r_{12} + r_{13}cos\theta_1 sin\theta_2 & r_{11}sin\theta_1 cos\theta_2 + r_{12}cos\theta_1 + r_{13}sin\theta_1 sin\theta_2 & r_{13}l_1 cos\theta_2 - r_{11}l_1 sin\theta_2 \\ r_{21}cos\theta_1 cos\theta_2 - r_{22}sin\theta_1 + r_{23}cos\theta_1 sin\theta_2 & r_{21}sin\theta_1 cos\theta_2 + cos\theta_1 r_{22} + r_{23}sin\theta_1 sin\theta_2 & r_{23}l_1 cos\theta_2 - r_{21}l_1 sin\theta_2 \\ -r_{32}cos\theta_1 cos\theta_2 - r_{32}sin\theta_1 + r_{33}cos\theta_1 sin\theta_2 & r_{31}sin\theta_1 cos\theta_2 + r_{32}cos\theta_1 + r_{33}sin\theta_1 sin\theta_2 & r_{33}l_1 cos\theta_2 - r_{31}l_1 sin\theta_2 \end{bmatrix}$$

Setting this R_{46} to the one in Equation 3 will give us expressions for θ_4, θ_5, and θ_6.

Problem 2: The hybrid Jacobian can be computed by first calculating the forward kinematics.

$$g_{ST} = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} cos\theta_1 cos(\theta_2+\theta_3) & -sin\theta_1 & cos\theta_1 sin(\theta_2+\theta_3) & cos\theta_1[l_2 cos\theta_2 + l_3 cos(\theta_2+\theta_3)] \\ sin\theta_1 cos(\theta_2+\theta_3) & cos\theta_1 & sin\theta_1 sin(\theta_2+\theta_3) & sin\theta_1[l_2 cos\theta_2 + l_3 cos(\theta_2+\theta_3)] \\ -sin(\theta_2+\theta_3) & 0 & cos(\theta_2+\theta_3) & -l_2 sin\theta_2 - l_3 sin(\theta_2+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Hybrid Jacobian is defined as

$$J^H = \begin{bmatrix} \frac{dp}{d\theta} \\ (\frac{dR}{d\theta}R^(-1))^V \end{bmatrix}$$

$$J^H = \begin{bmatrix} -sin\theta_1[l_2 cos\theta_2 + l_3 cos(\theta_2+\theta_3)] & -cos\theta_1[l_2 sin\theta_2 + l_3 sin(\theta_2+\theta_3)] & -l_3 cos\theta_1 sin(\theta_2+\theta_3) \\ cos\theta_1[l_2 cos\theta_2 + l_3 cos(\theta_2+\theta_3)] & -sin\theta_1[l_2 sin\theta_2 + l_3 sin(\theta_2+\theta_3)] & -l_3 sin\theta_1 sin(\theta_2+\theta_3) \\ 0 & -l_2 cos\theta_2 - l_3 cos(\theta_2+\theta_3) & -l_3 cos(\theta_2+\theta_3) \\ 0 & -sin\theta_1 & -sin\theta_1 \\ 0 & cos\theta_1 & cos\theta_1 \\ 1 & 0 & 0 \end{bmatrix}$$

There exist singularities when the determinant of J equals zero.

$$0 = det[{J^H}^T J^H] = -l_2 l_3 (l_2 cos\theta_2 + l_3 cos(\theta_2 + \theta_3)) sin\theta_3 \tag{5}$$

This corresponds to two cases. The first is when $\theta_3 = 0$ and θ_2 is anything. The other case is when $\theta_2 = \frac{\pi}{2}$ andθ_3=π.