ME 115(b): Solution to Homework #1

Solution to Problem #1:

To construct the hybrid Jacobian for a manipulator, you could either construct the body Jacobian, J^b_{ST}, and then use the body-to-hybrid velocity transformation:

$$J^h_{ST} = \begin{bmatrix} R_{ST} & 0 \\ 0 & R_{ST} \end{bmatrix} J^b_{ST}$$

or recall that the columns of the hybrid Jacobian take the form:

$$J^h_{ST} = \begin{bmatrix} \frac{\partial \vec{p}_{ST}}{\partial \theta_1} & \frac{\partial \vec{p}_{ST}}{\partial \theta_2} & \cdots & \frac{\partial \vec{p}_{ST}}{\partial \theta_N} \\ & & & \\ \vec{\omega}_1 & \vec{\omega}_2 & \cdots & \vec{\omega}_N \end{bmatrix} \quad (1)$$

where the forward kinematics equations $g_{ST}(\vec{\theta})$ take the form:

$$g_{ST}(\vec{\theta}) = \begin{bmatrix} R_{ST}(\vec{\theta}) & \vec{p}_{ST}(\vec{\theta}) \\ \vec{0}^T & 1 \end{bmatrix}$$

and $\vec{\omega}_j$ is:

$$\vec{\omega}_j = \left(\frac{\partial R_{ST}}{\partial \theta_j} R^T_{ST} \right)^{\vee} .$$

This solution will use the second approach.

Manipulator (ii): While this manipulator has a rather odd geometry, it is relatively straightforward to tackle this problem by suitable choices of geometry in the Denavit-Hartenburg approach. If β is the angle between the first and third joint axes when the manipulator lies in the configuration shown in the book, and if l_1 and l_2 are the two link lengths as shown in the book's figure, then the D-H parameters for this manipulator are:

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = l_1 \cos\beta & \alpha_2 = -\frac{\pi}{2} & d_3 = (l_1 + l_2)\sin\beta & \theta_3 = \text{ variable} \\ a_3 = l_2 \cos\beta & \alpha_3 = 0 & d_4 = 0 & \theta_4 = 0 \end{array}$$

where we have assumed that the tool frame z-axis is parallel to joint axis 3. Recalling the the relationship between link frames in terms of the D-H parameters:

$$g_{i,i+1} = \begin{bmatrix} \cos\theta_{i+1} & -\sin\theta_{i+1} & 0 & a_i \\ \sin\theta_{i+1}\cos\alpha_i & \cos\theta_{i+1}\cos\alpha_i & -\sin\alpha_i & -d_{i+1}\sin\alpha_i \\ \sin\theta_{i+1}\sin\alpha_i & \cos\theta_{i+1}\sin\alpha_i & \cos\alpha_i & d_{i+1}\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The forward kinematics of this mechanism can be found as:

$$
\begin{aligned}
g_{ST} &= g_{S,1}\; g_{1,2}\; g_{2,3}\; g_{3,T} \\
&= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ 0 & 0 & -1 & -d_3 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2) \\
&= \begin{bmatrix} (c_1c_2c_3 - s_1s_3) & -(c_1c_2s_3 + s_1c_3) & c_1s_2 & (a_3c_1c_2c_3 + a_2c_1c_2 + d_3c_1s_2 - a_3s_1s_3) \\ (s_1c_2c_3 + c_1s_3) & (-s_1c_2s_3 + c_1c_3) & s_1s_2 & (a_3s_1c_2c_3 + a_2s_1c_2 + d_3s_1s_2 + a_3c_1s_3) \\ -s_2c_3 & s_2s_3 & c_2 & (-a_3s_2c_3 - a_2s_2 + d_3c_2) \end{bmatrix}
\end{aligned}
$$

Following Equation (1), the hybrid Jacobian is:

$$
\begin{aligned}
J^h_{ST} &= \begin{bmatrix} \frac{\partial \vec{p}_{ST}}{\partial \theta_1} & \frac{\partial \vec{p}_{ST}}{\partial \theta_2} & \frac{\partial \vec{p}_{ST}}{\partial \theta_3} \\ \left(\frac{\partial R_{ST}}{\partial \theta_1} R^T_{ST}\right)^\vee & \left(\frac{\partial R_{ST}}{\partial \theta_2} R^T_{ST}\right)^\vee & \left(\frac{\partial R_{ST}}{\partial \theta_2} R^T_{ST}\right)^\vee \end{bmatrix} \qquad (3) \\
&= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vec{\omega}_1 & \vec{\omega}_2 & \vec{\omega}_3 \end{bmatrix}
\end{aligned}
$$

where:

$$
\vec{v}_1 = \begin{bmatrix} (-a_3s_1c_2c_3 - a_2s_1c_2 - d_3s_1s_2 - a_3c_1s_3) \\ (a_3c_1c_2c_3 + a_2c_1c_2 + d_3c_1s_2 - a_3s_1s_3) \\ 0 \end{bmatrix}
$$

$$
\vec{v}_2 = \begin{bmatrix} (-a_3c_1s_2c_3 - a_2c_1s_2 + d_3c_1c_2) \\ (-a_3s_1s_2c_3 - a_2s_1s_2 + d_3s_1c_2) \\ (-a_3c_2c_3 - a_2c_2 - d_3s_2) \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} (-a_3c_1c_2s_3 - a_3s_1c_3) \\ (-a_2s_1c_2s_3 + a_3c_1c_3) \\ (a_3s_2s_3) \end{bmatrix}
$$

$$
\begin{bmatrix} \vec{\omega}_1 & \vec{\omega}_2 & \vec{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 & -s_1 & c_1s_2 \\ 0 & c_1 & s_1s_2 \\ 1 & 0 & c_2 \end{bmatrix}
$$

Manipulator (iv): This is the "Stanford Manipulator" that we analyzed in class, and that you also analyzed in Homework #4 of ME 115(a). From that homework solution, recall that the Denavit-Hartenberg parameters and the forward kinematics are:

$$
\begin{array}{llll}
a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{ variable} \\
a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\
a_2 = 0 & \alpha_2 = -\frac{\pi}{2} & d_3 = \text{variable} & \theta_3 = 0 \text{ (constant)}
\end{array}
$$

$$
\begin{aligned}
g_{ST} &= g_{S,1}g_{1,2}g_{2,3}g_{3,T} && (4)\\
&= \begin{bmatrix} c_1 & -s_1 & 0 & 0\\ s_1 & c_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ -s_2 & -c_2 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & -1 & -d_3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} && (5)\\
&= \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2\\ s_1c_2 & c_1 & s_1s_2 & d_3s_1s_2\\ -s_2 & 0 & c_2 & d_3c_2\\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ST}(\theta_1,\theta_2,d_3) & \vec{p}_{ST}(\theta_1,\theta_2,d_3)\\ \vec{0}^T & 1 \end{bmatrix} && (6)
\end{aligned}
$$

Following Equation (1), the hybrid Jacobian is:

$$
\begin{aligned}
J^h_{ST} &= \begin{bmatrix} \frac{\partial \vec{p}_{ST}}{\partial \theta_1} & \frac{\partial \vec{p}_{ST}}{\partial \theta_2} & \frac{\partial \vec{p}_{ST}}{\partial d_3}\\ \left(\frac{\partial R_{ST}}{\partial \theta_1}R^T_{ST}\right)^\vee & \left(\frac{\partial R_{ST}}{\partial \theta_2}R^T_{ST}\right)^\vee & \left(\frac{\partial R_{ST}}{\partial \theta_2}R^T_{ST}\right)^\vee \end{bmatrix} && (7)\\
& && (8)\\
&= \begin{bmatrix} -d_3s_1s_2 & d_3c_1c_2 & c_1s_2\\ d_3c_2s_2 & d_3s_1c_2 & s_1s_2\\ 0 & -d_3s_2 & c_2\\ 0 & -s_1 & 0\\ 0 & c_1 & 0\\ 1 & 0 & 0 \end{bmatrix} && (9)
\end{aligned}
$$

Solution to Problem #2:

To find the singularities of the *regional* part (just the first three joints) of the elbow manipulator, one can determine the conditions under which the Jacobian matrix of the manipulator loses rank. While one could use any Jacobian, for simplicity we will use the Hybrid Jacobian matrix. You could either recall from the class note (or derive) the forward kinematics of the Elbow manipulator:

$$
\begin{aligned}
g_{ST}(\vec{\theta}) &= \begin{bmatrix} R_{ST}(\vec{\theta}) & p_{ST}(\vec{\theta})\\ \vec{0}^T & 1 \end{bmatrix}\\
&= \begin{bmatrix} c_1c_{23} & -s_1 & c_1s_{23} & c_1(l_2c_2+l_3c_{23})\\ s_1c_{23} & c_1 & s_1s_{23} & s_1(l_2c_2+l_3c_{23})\\ -s_{23} & 0 & c_{23} & -(l_2s_2+l_3s_{23})\\ 0 & 0 & 0 & 1 \end{bmatrix} && (10)
\end{aligned}
$$

where $c_j = \cos(\theta_j)$, $s_j = \sin(\theta_j)$, $c_{ij} = \cos(\theta_i + \theta_j)$, $s_{ij} = \sin(\theta_i + \theta_j)$, etc. Recall that the hybrid Jacobian for the regional part of a manipulator is defined as:

$$
J^h_{ST} = \begin{bmatrix} \frac{\partial \vec{p}_{ST}}{\partial \vec{\theta}} \end{bmatrix} \qquad (11)
$$

and thus substituting Equation (10) into Equation (11) yields:

$$J_{ST}^h = \begin{bmatrix} -s_1(l_2c_2 + l_3c_{23}) & -c_1(l_2s_2 + l_3s_{23}) & -l_3c_1s_{23} \\ c_1(l_2c_2 + l_3c_{23}) & -s_1(l_2s_2 + l_3s_{23}) & -l_3s_1s_{23} \\ 0 & -(l_2c_2 + l_3c_{23}) & -l_3c_{23} \end{bmatrix}. \tag{12}$$

Singularities will occur when the determinant of J_{ST}^h is zero:

$$det(J_{ST}^h) = -l_2l_3[l_2\cos(\theta_2) + l_3\cos(\theta_2+\theta_3)]\sin(\theta_3)$$

The singularities occur when:

- $\theta_3 = 0$. In this case, the arm is fully "streteched out", and this singular configuration corresponds to the manipulator's outer workspace limit.
- $\theta_3 = \pm\pi$. In this case, the arm is folded back on itself, and this singular configuration corresponds to the manipulator's inner workspace boundary.
- $l_2c_2 + l_3c_{23} = 0$. Note from the forward kinematics equations that in this case, x and y coordinates of the tool frame origin lie at $x = 0$ and $y = 0$. This occurs when the tool frame origin is placed anywhere along the first joint axis.