ME 115(a): Solution to Homework #1

Problem 1: Problem 12, Chapter 2 of MLS

We know from Lemma 2.3 that:

$$\hat{\omega}^3 = -||\omega||^2\hat{\omega}$$

$$\hat{\omega}^4 = -||\omega||^2\hat{\omega^2}$$

$$\hat{\omega}^5 = ||\omega||^4\hat{\omega}$$

$$\hat{\omega}^6 = ||\omega||^4\hat{\omega}^2$$

$$\hat{\omega}^7 = -||\omega||^6\hat{\omega}$$

$$\hat{\omega}^8 = -||\omega||^6\hat{\omega}^2$$

Plugging these values into the matrix exponential:

$$\begin{aligned}
e^{\hat{\omega}\theta} &= I + \hat{\omega}\theta + \tfrac{(\hat{\omega}\theta)^2}{2!} + \tfrac{(\omega\theta)^3}{3!} + \cdots \\
&= I + (\theta\hat{\omega} + \tfrac{(\theta\hat{\omega})^3}{3!} + \cdots) + (\tfrac{(\theta\hat{\omega})^2}{2!} + \tfrac{(\theta\hat{\omega})^4}{4!} + \cdots) \\
&= I + (\theta||\omega|| - \tfrac{(\theta||\omega||)^3}{3!} + \cdots)\tfrac{\hat{\omega}}{||\omega||} + (\tfrac{(\theta||\omega||)^2}{2!} - \tfrac{(\theta||\omega||)^4}{4!} + \cdots)\tfrac{\hat{\omega}^2}{||\omega||^2} \\
&= I + sin(||\omega||\theta)\tfrac{\hat{\omega}}{||\omega||} + (1 - cos(||\omega||\theta)\tfrac{\hat{\omega}^2}{||\omega||^2}
\end{aligned}$$

Problem 2: Problem 10(a,b), Chapter 2 of MLS

- **Part (a):** Let:

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $A \in$ SO(2). Thus det(A) = 1, and $A^{-1} = A^T$.

$$A^T A = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{11}a_{21} + a_{21}a_{22} & a_{12}^2 + a_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving the above equations, we find that $a_{21} = -a_{12}$, and $a_{22} = a_{11}$. Given that det(A) =1, we know that $a_{11}a_{22} - a_{21}a_{12} = 1$. Setting $a_{11} = cos\theta, a_{12} = sin\theta, a_{21} = sin\theta$, and $a_{22} = cos\theta$ meets these requirements.

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Both vectors $\vec{a}_1$ and $\vec{a}$ are elements of S^1, the unit circle.

- **Part (b):** If $\omega \in \mathbb{R}$, then let:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \omega J \qquad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that:

$$\hat{\omega}^2 = -\omega^2 I; \qquad \hat{\omega}^3 = \omega^3 J$$

Hence:

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \omega\theta J + \tfrac{(\omega\theta)^2}{2!} J^2 + \tfrac{(\omega\theta)^3}{3!} J^3 + \cdots \\ &= I + (\omega\theta) J - \tfrac{(\omega\theta)^2}{2!} I - \tfrac{(\omega\theta)^3}{3!} J + \cdots \\ &= (1 + \tfrac{(\omega\theta)^2}{2!} + \tfrac{(\omega\theta)}{4!} + \cdots) I + (\omega\theta - \tfrac{(\omega\theta)^3}{3!} + \cdots) J \\ &= \cos(\omega\theta) I + \sin(\omega\theta) J \\ &= \begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) \\ \sin(\omega\theta) & \cos(\omega\theta) \end{bmatrix} \end{aligned}$$

Problem 3: Problem 8(b), Chapter 2 of MLS

$$\begin{aligned} e^{g\Lambda g^{-1}} &= I + \frac{1}{1!} g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\ &= I + \frac{1}{1!} g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\ &= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots) g^{-1} \\ &= g e^{\Lambda} g^{-1} \end{aligned}$$

Problem 4: Problem 5, Chapter 2 of MLS

- **Part (a):**

$$\begin{aligned} R_a R_a^T &= (I - \hat{a})^{-1}(I + \hat{a})(I + \hat{a})^T (I - \hat{a})^{-T} \\ &= (I - \hat{a})^{-1}(I + \hat{a})(I - \hat{a})(I + \hat{a})^{-1} \\ &= (I - \hat{a})^{-1}(I - \hat{a})(I + \hat{a})(I + \hat{a})^{-1} \\ &= I * I \\ &= I \end{aligned}$$

This means that det $(R_a) = 1$, and thays $R_a \in \mathrm{SO}(3)$.

- **Part (b):** This part can be done, by hand or using Mathematica, by simply expanding out the equation from part (a):

$$R_a = (I - \hat{a})^{-1}(I + \hat{a}).$$

- **Part (c):** There are two ways to solve this. The simplest way is to use the result of part 5(b) quoted in the text:

$$R = \frac{1}{1+||a||^2}\begin{bmatrix} 1+a_1^2-a_2^2-a_3^2 & 2(a_1a_2-a_3) & 2(a_1a_3+a_2) \\ 2(a_1a_2+a_3) & 1-a_1^2+a_2^2-a_3^2 & 2(a_2a_3-a_1) \\ 2(a_1a_3-a_2) & 2(a_2a_3+a_1) & 1-a_1^2-a_2^2+a_3^2 \end{bmatrix} \tag{1}$$

where $||a||^2$ is shorthand notation for $||a||^2 = a_1^2 + a_2^2 + a_3^2$. Noting that

$$trace(R) = \frac{3-||a||^2}{1+||a||^2} \Rightarrow ||a||^2 = \frac{3-trace(R)}{1+trace(R)} = \frac{3-r_{11}-r_{22}-r_{33}}{1+r_{11}+r_{22}+r_{33}}$$

so that an expression for $||a||^2$ is known, simple algebraic manipulation of the off-diagonal term of R yield

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1+||a||^2}{4}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

Problem 5:

To show that $\mathrm{cofactor}(\mathrm{r_{ii}}) = \mathrm{r_{ii}}$, let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

The columns of this matrix are unit vectors, which can be denoted as:

$$\mathbf{x} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$$

These columns can be interpreted as the unit vectors of an orthogonal right handed coordinate system. Consequently,

$$\mathbf{x} = \mathbf{y} \times \mathbf{z}; \qquad \mathbf{y} = \mathbf{z} \times \mathbf{x}; \qquad \mathbf{z} = \mathbf{x} \times \mathbf{y}$$

Performing the cross product and equating sides for $\mathbf{x} = \mathbf{y} \times \mathbf{z}$, we get the relation:

$$\begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} = \begin{bmatrix} r_{22}r_{33}-r_{23}r_{32} \\ r_{13}r_{32}-r_{12}r_{33} \\ r_{12}r_{23}-r_{13}r_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{cofactor}(\mathrm{r_{11}}) \\ \mathrm{cofactor}(\mathrm{r_{21}}) \\ \mathrm{cofactor}(\mathrm{r_{31}}) \end{bmatrix} \tag{2}$$

Similar relationships can be derived for the other columns to show that $r_{ij} = \text{cofactor}(r_{ij})$ for all elements of a special orthogonal matrix.

Problem 6: Problem 4(a,b) in Chapter 2 of MLS

- **Part (a):** Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\ &= R(\vec{w} \times (R^T\vec{v})) \\ &= (R\vec{w}) \times (RR^T\vec{v}) \\ &= (R\vec{w}) \times \vec{v} \\ &= \widehat{(R\vec{w})}\vec{v}\end{aligned}$$

  Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

- **Part (b):** We can now assume that part (a) holds.

$$\begin{aligned}(R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\ &= (R\hat{v}R^T)(R\vec{w}) \\ &= R\hat{v}R^TR\vec{w} \\ &= R(\hat{v}\vec{w}) \\ &= R(\vec{v} \times \vec{w})\end{aligned}$$

Problem 7:
Find the axis of rotation and angle of rotation associated with the rotation matrix:

$$\begin{bmatrix} 0.866025 & -0.353553 & 0.353553 \\ 0.353553 & 0.933013 & 0.0669873 \\ -0.353553 & 0.0669873 & 0.933013 \end{bmatrix}$$

From Eq. (2.17) in the MLS text:

$$\cos(\phi) = \frac{r_{11} + r_{22} + r_{33} - 1}{2} = \frac{0.866025 + 0.933013 + 0.933013 - 1.0}{2} = 0.866$$

Thus, $\phi = \cos^{-1}(0.866) = 30^o$. Thus, $\sin(\phi) = 0.5$, and therefore from Eq. (2.18) of the MLS text:

$$\begin{aligned}\omega_x &= \tfrac{r_{32} - r_{23}}{2\sin\phi} = 0.0 \\ \omega_y &= \tfrac{r_{13} - r_{31}}{2\sin\phi} = 0.7071 \\ \omega_z &= \tfrac{r_{21} - r_{12}}{2\sin\phi} = 0.7071\end{aligned}$$