Problem 1: Find the axis of rotation and angle of rotation associated with the following rotation matrix:

$$
\begin{bmatrix}
0.866025 & -0.353553 & 0.353553 \\
0.353553 & 0.933013 & 0.0669873 \\
-0.353553 & 0.0669873 & 0.933013
\end{bmatrix}.
$$

Let the matrix entries be denoted $a_{ij}$, where $i = 1, 2, 3$ denotes the row, and $j = 1, 2, 3$ denotes the column. From class notes or the text:

$$
cos(\phi) = \frac{a_{11} + a_{22} + a_{33} - 1}{2} = \frac{0.866025 + 0.933013 + 0.933013 - 1.0}{2} = 0.866025
$$

where $\phi$ is the angle of rotation. Thus, $\phi = \cos^{-1}(0.866025) = 30^\circ$. Thus, $\sin(\phi) = 0.5$, and therefore:

$$
s_x = \frac{a_{32} - a_{23}}{2 \sin \phi} = 0.0 \quad (1)
$$

$$
s_y = \frac{a_{13} - a_{31}}{2 \sin \phi} = 0.707106 \quad (2)
$$

$$
s_z = \frac{a_{21} - a_{12}}{2 \sin \phi} = 0.707106 \quad (3)
$$

Problem 2: Can every orthogonal matrix be represented by the exponential of a real matrix?

You should have either remembered or derived the fact that $\det(e^C) = e^{\text{tr}(C)}$, where $\text{tr}(C)$ is the trace of determinant $C$. Note that if $\text{tr}(C)$ is real, than $e^{\text{tr}(C)}$ is always a positive number and therefore orthogonal matrices with determinant -1 can not be represented as a matrix exponential. This arises from the fact that the Orthogonal Group is a disconnected group. That is, the matrices with +1 determinant are all connected to each other, and similarly the ones with -1 determinant. But, the two subsets are disjoint.

Note, that if $\text{tr}(C) = i\pi$ (where $i^2 = -1$), then $\det(e^C) = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$. However, $\text{tr}(C)$ can not assume the value of $i\pi$ if $C$ is real. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Let $C$ be a $n \times n$ matrix. If $n$ is even, then all of the eigenvalues of $C$ must be complex conjugates, or an even number of real roots. Thus, the sum of the eigenvalues must be real. Similarly, if $n$ is odd, the eigenvalues will either be complex conjugates and/or an odd number of real eigenvalues. Thus, the sum of the eigenvalues must also be real number. Thus, if $C$ is a real matrix (as specified in part (a)), then $e^C$ can not represent orthogonal matrices with determinant -1.

Problem 3: Let Z-X-Y Euler angles be denoted by $\psi$, $\phi$, and $\gamma$. 

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• **Part (a):** Develop an expression for the rotation matrix that describes the Z-X-Y rotation as a function of the angles $\psi$, $\phi$, and $\gamma$.

Rotation about the $z$-axis by angle $\psi$ can be represented by a rotation matrix whose form can be determined from the Rodriguez Equation:

$$\text{Rot}(\vec{z}, \psi) = I + \sin \psi \hat{z} + (1 - \cos \psi) \hat{z}^2 = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

Using the Rodriguez equation, the rotations about the $y$-axis and $x$-axis can be similarly found as:

$$\text{Rot}(\vec{x}, \phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}, \quad \text{Rot}(\vec{y}, \gamma) = \begin{bmatrix} \cos \gamma & 0 & \sin \gamma \\ 0 & 1 & 0 \\ -\sin \gamma & 0 & \cos \gamma \end{bmatrix}.$$  

Multiplying the matrices yields the result:

$$R(\psi, \phi, \gamma) = \text{Rot}(\vec{z}, \psi) \text{ Rot}(\vec{x}, \phi) \text{ Rot}(\vec{y}, \gamma) = \begin{bmatrix} c\psi c\gamma - s\psi s\phi s\gamma & -s\psi c\phi & (c\psi s\gamma + s\psi s\phi c\gamma) \\ (c\psi c\gamma + s\psi s\phi s\gamma) & c\psi c\phi & (s\psi s\gamma - c\psi s\phi c\gamma) \\ -s\phi s\gamma & s\phi & c\phi c\gamma \end{bmatrix}.$$  

where $c\phi$ and $s\phi$ are respectively shorthand notation for $\cos \phi$ and $\sin \phi$, etc.

• **Part (b):** Given a rotation matrix of the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$  

compute the angles $\psi$, $\phi$, and $\gamma$ as a function of the $r_{ij}$.

Direct observation of the matrices in Equations (4) and (5) show that:

$$\sin \phi = r_{32}.$$  

Because $\sin(\pi - \phi) = \sin \phi$, there are two solutions to this equation: $\phi_1 = \sin^{-1}(r_{32})$, and $\phi_2 = \pi - \phi_1$. Similar matchings of the matrix components yield:

$$\psi = \text{Atan}2\left[ \frac{r_{22}}{\cos \phi}, -\frac{r_{12}}{\cos \phi} \right],$$

$$\gamma = \text{Atan}2\left[ \frac{r_{33}}{\cos \phi}, -\frac{r_{31}}{\cos \phi} \right],$$

where the value $\phi_1$ or $\phi_2$ is used consistently.
Problem 4: Problem 4(a,b) in Chapter 2 of MLS

Part (a): Let’s assume that the statement in part (b) of the problem is true. Let \( \vec{w} \) be a \( 3 \times 1 \) vector and let \( \vec{v} \) be any \( 3 \times 1 \) vector. Then:

\[
(R \hat{\vec{w}} R^T) \vec{v} = R(\hat{\vec{w}} \times (R^T \vec{v})) = (R \hat{\vec{w}}) \times (RR^T \vec{v}) = (R \hat{\vec{w}}) \times \vec{v} = (R \hat{\vec{w}}) \hat{\vec{v}}
\]

Since this must be true for any vector \( \vec{v} \), then \( R \hat{\vec{w}} R^T = (R \hat{\vec{w}}) \).

Part (b): We can now assume that part (a) holds.

\[
(R \vec{v}) \times (R \hat{\vec{w}}) = (R \vec{v})(R \hat{\vec{w}}) = (R \hat{\vec{w}})(R \vec{v}) = R \hat{\vec{w}} R^T \hat{\vec{w}} = R(\hat{\vec{w}} \vec{v}) = R(\hat{\vec{w}} \times \vec{v})
\]

Problem 5: Problem 5 in Chapter 2 of MLS

Part (a): To show that a matrix, \( R \) is in \( SO(3) \), we must show that \( R R^T = I \), where \( I \) is the \( 3 \times 3 \) identity matrix, and that \( \det(R) = 1 \). We are given that:

\[
R = (I - \hat{a})^{-1}(I + \hat{a})
\]

where \( \hat{a} \) is a \( 3 \times 3 \) skew symmetric matrix. Thus, the first step is:

\[
R R^T = (I - \hat{a})^{-1}(I + \hat{a})(I - \hat{a})^T = (I - \hat{a})^{-1}(I + \hat{a})(I - \hat{a})(I - \hat{a})^{-1}
\]

where we have used the fact that \( \hat{a}^T = -\hat{a} \) in the second identity. Note that \( (I + \hat{a})(I - \hat{a}) = I - \hat{a}^2 = (I - \hat{a})(I + \hat{a}) \). Substituting this result into Equation (7) yields the answer:

\[
R R^T = (I - \hat{a})^{-1}(I + \hat{a})(I - \hat{a})(I + \hat{a})^{-1} = (I - \hat{a})^{-1}(I - \hat{a})(I + \hat{a})(I + \hat{a})^{-1} = I
\]

To check the determinant, note that

\[
\det(R) = \det((I - \hat{a})^{-1}(I + \hat{a})) = \frac{\det(I + \hat{a})}{\det(I - \hat{a})}.
\]
We have already shown that $R$ is an orthogonal matrix, and therefore its determinant is +1 or −1. Since the determinant is a continuous function, at $a = 0$ note that $\det(R) = +1$. Hence, $\det(R) = +1$ for all values of $\hat{a}$. If you didn’t know that $\det(R)$ is a continuous function, then brute force algebra will yield the same result.

**Part (b):** There were several ways to show that $R = (I - \hat{a})^{-1}(I + \hat{a}) = \frac{1}{1 + ||a||^2} \begin{bmatrix} 1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_3) & 2(a_1a_3 + a_2) \\ 2(a_1a_2 + a_3) & 1 - a_2^2 + a_3^2 & 2(a_2a_3 - a_1) \\ 2(a_1a_3 - a_2) & 2(a_2a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix}$ (12)

Of course, you could have expanded out the expression for $R$ to show the equivalence. The least grungy way (in terms of messy algebra) was to show that $(I - \hat{a})R = I + \hat{a}$, where $R$ is the expression for the rotation matrix in terms of $a_1$, $a_2$, and $a_3$ (Equation (13) below).

**Part (c):** There are multiple ways to solve this problem. The simplest way is to use the expression of part 5(b) (Equation (12) quoted in the text):

$$R = \frac{1}{1 + ||a||^2} \begin{bmatrix} 1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_3) & 2(a_1a_3 + a_2) \\ 2(a_1a_2 + a_3) & 1 - a_2^2 + a_3^2 & 2(a_2a_3 - a_1) \\ 2(a_1a_3 - a_2) & 2(a_2a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix}$$ (13)

where $||a||^2$ is shorthand notation for $||a||^2 = a_1^2 + a_2^2 + a_3^2$. Noting that

$$\text{trace}(R) = \frac{3 - ||a||^2}{1 + ||a||^2} \Rightarrow ||a||^2 = \frac{3 - \text{trace}(R)}{1 + \text{trace}(R)} = \frac{3 - r_{11} - r_{22} - r_{33}}{1 + r_{11} + r_{22} + r_{33}}$$

so that an expression for $||a||^2$ is known, simple algebraic manipulation of the off-diagonal term of $R$ in Equation (13) yield

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1 + ||a||^2}{4} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

**Problem 6:** Problem 10 (a,b) in Chapter 2 of MLS (not including the question of surjectivity in 10(b)).

Note that

$$\dot{\hat{\omega}} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$\dot{\hat{\omega}}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2I; \quad \dot{\hat{\omega}}^3 = -w^3J$$
Hence the exponential of $\hat{\omega}$ can be computed as:

$$
\exp (\theta \hat{\omega}) = \left( I + \frac{\theta}{1!} \hat{\omega} + \frac{\theta^2}{2!} \hat{\omega}^2 + \ldots \right)
= \left( I + \frac{w \theta}{1!} J - \frac{w^2 \theta^2}{2!} I - \frac{w^3 \theta^3}{3!} J + \ldots \right)
= \left( 1 - \frac{w^2 \theta^2}{2!} + \ldots \right) I + \left( \frac{w \theta}{1!} - \frac{w^3 \theta^3}{3!} + \ldots \right) J
= \begin{bmatrix}
\cos(w \theta) & -\sin(w \theta) \\
\sin(w \theta) & \cos(w \theta)
\end{bmatrix}
$$