ME 115(a): Solution to Homework #2

Problem 1: Find the axis of rotation and angle of rotation associated with the following rotation matrix:

$$\begin{bmatrix} 0.866025 & -0.353553 & 0.353553 \\ 0.353553 & 0.933013 & 0.0669873 \\ -0.353553 & 0.0669873 & 0.933013 \end{bmatrix}.$$

Let the matrix entries be denoted a_{ij}, where $i = 1, 2, 3$ denotes the row, and $j = 1, 2, 3$ denotes the column. From class notes or the text:

$$\cos(\phi) = \frac{a_{11} + a_{22} + a_{33} - 1}{2} = \frac{0.866025 + 0.933013 + 0.933013 - 1.0}{2} = 0.8660245$$

where ϕ is the angle of rotation. Thus, $\phi = \cos^{-1}(0.8660245) = 30^o$. Thus, $\sin(\phi) = 0.5$, and therefore:

$$\begin{aligned} s_x &= \frac{a_{32} - a_{23}}{2\sin\phi} = 0.0 && (1) \\ s_y &= \frac{a_{13} - a_{31}}{2\sin\phi} = 0.707106 && (2) \\ s_z &= \frac{a_{21} - a_{12}}{2\sin\phi} = 0.707106 && (3) \end{aligned}$$

Problem 2: Can every orthogonal matrix be represented by the exponential of a real matrix?

You should have either remembered or derived the fact that $det(e^C) = e^{tr(C)}$, where $tr(C)$ is the trace of determinant C. Note that if $tr(C)$ is real, than $e^{tr(C)}$ is always a positive number and therefore orthogonal matrices with determinant -1 can not be represented as a matrix exponential. This arises from the fact that the Orthogonal Group is a disconnected group. That is, the matrices with +1 determinant are all connected to each other, and similarly the ones with -1 determinant. But, the two subsets are disjoint.

Note, that if $tr(C) = i\pi$ (where $i^2 = -1$), then $det(e^C) = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$. However, $tr(C)$ can not assume the value of $i\pi$ if C is real. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Let C be a $n \times n$ matrix. If n is even, then all of the eigenvalues of C must be complex conjugates, or an even number of real roots. Thus, the sum of the eigenvalues must be real. Similarly, if n is odd, the eigenvalues will either be complex conjugates and/or an odd number of real eigenvalues. Thus, the sum of the eigenvalues must also be real number. Thus, if C is a real matrix (as specified in part (a)), then e^C can not represent orthogonal matrices with determinant -1.

Problem 3: Let Z-X-Y Euler angles be denoted by ψ, ϕ, and γ.

- **Part (a):** Develop an expression for the rotation matrix that describes the Z-X-Y rotation as a function of the angles ψ, ϕ, and γ.

  Rotation about the z-axis by angle ψ can be represented by a rotation matrix whose form can be determined from the Rodriguez Equation:

$$Rot(\vec{z}, \psi) = I + \sin\psi \hat{z} + (1 - \cos\psi)\hat{z}^2 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

  Using the Rodriguez equation, the rotations about the y-axis and x-axis can be similarly found as:

$$Rot(\vec{x}, \phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad Rot(\vec{y}, \gamma) = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} .$$

  Multiplying the matrices yields the result:

$$\begin{aligned} R(\psi, \phi, \gamma) &= Rot(\vec{z}, \psi)\ Rot(\vec{x}, \phi)\ Rot(\vec{y}, \gamma) \\ &= \begin{bmatrix} (c\psi\ c\gamma\ -\ s\psi\ s\phi\ s\gamma) & -s\psi\ c\phi & (c\psi\ s\gamma\ \ \ \ (c\psi\ s\gamma\ +\ s\psi\ s\phi\ c\gamma) \\ (s\psi\ c\gamma\ +\ c\psi\ s\phi\ s\gamma) & c\psi\ c\phi & (s\psi\ s\gamma\ -\ c\psi\ s\phi\ c\gamma) \\ -c\phi\ s\gamma & s\phi & c\phi\ c\gamma \end{bmatrix} \end{aligned} \tag{4}$$

  where $c\phi$ and $s\phi$ are respectively shorthand notation for $\cos\phi$ and $\sin\phi$, etc.

- **Part (b):** Given a rotation matrix of the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{5}$$

  compute the angles ψ, ϕ, and γ as a function of the r_{ij}.

  Direct observation of the matrices in Equations (4) and (5) show that:

$$\sin\phi = r_{32} .$$

  Because $\sin(\pi - \phi) = \sin\phi$, there are two solutions to this equation: $\phi_1 = \sin^{-1}(r_{32})$, and $\phi_2 = \pi - \phi_1$. Similar matchings of the matrix components yield:

$$\psi = Atan2[\frac{r_{22}}{\cos\phi}, \frac{-r_{12}}{\cos\phi}]$$

$$\gamma = Atan2[\frac{r_{33}}{\cos\phi}, \frac{-r_{31}}{\cos\phi}]$$

  where the value ϕ_1 or ϕ_2 is used consistently

Problem 4: Problem 4(a,b) in Chapter 2 of MLS

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}
(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\
&= R(\vec{w} \times (R^T\vec{v})) \\
&= (R\vec{w}) \times (RR^T\vec{v}) \\
&= (R\vec{w}) \times \vec{v} \\
&= \widehat{(R\vec{w})}\vec{v}
\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}
(R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\
&= (R\hat{v}R^T)(R\vec{w}) \\
&= R\hat{v}R^TR\vec{w} \\
&= R(\hat{v}\vec{w}) \\
&= R(\vec{v} \times \vec{w})
\end{aligned}$$

Problem 5: Problem 5 in Chapter 2 of MLS

Part (a): To show that a matrix, R is in $SO(3)$, we must show that $R\ R^T = I$, where I is the 3×3 identity matrix, and that $det(R) = 1$. We are given that:

$$R = (I - \hat{a})^{-1}(I + \hat{a}) \tag{6}$$

where $\hat{a}$ is a 3×3 skew symmetric matrix. Thus, the first step is:

$$\begin{aligned}
R\ R^T &= (I - \hat{a})^{-1}(I + \hat{a})(I + \hat{a})^T(I - \hat{a})^{-T} && (7) \\
&= (I - \hat{a})^{-1}(I + \hat{a})(I - \hat{a})(I + \hat{a})^{-1} && (8)
\end{aligned}$$

where we have used the fact that $\hat{a}^T = -\hat{a}$ in the second identity. Note that $(I + \hat{a})(I - \hat{a}) = I - \hat{a}^2 = (I - \hat{a})(I + \hat{a})$. Substituting this result into Equation (7) yields the answer:

$$\begin{aligned}
R\ R^T &= (I - \hat{a})^{-1}(I + \hat{a})(I - \hat{a})(I + \hat{a})^{-1} && (9) \\
&= (I - \hat{a})^{-1}(I - \hat{a})(I + \hat{a})(I + \hat{a})^{-1} && (10) \\
&= I && (11)
\end{aligned}$$

To check the determinant, note that

$$det(R) = det[(I - \hat{a})^{-1}(I + \hat{a})] = \frac{det(I + \hat{a})}{det(I - \hat{a})}\ .$$

We have already shown that R is an orthogonal matrix, and therefore it's determinant is $+1$ or -1. Since the determinant is a continuous function, at $a = 0$ note that $det(R) = +1$. Hence, $det(R) = +1$ for all values of $\hat{a}$. If you didn't know that $det(R)$ is a continuous function, then brute force algebra will yield the same result.

Part (b): There were several ways to show that

$$R = (I - \hat{a})^{-1}(I + \hat{a}) = \frac{1}{1 + ||a||^2} \begin{bmatrix} 1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_3) & 2(a_1a_3 + a_2) \\ 2(a_1a_2 + a_3) & 1 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_1) \\ 2(a_1a_3 - a_2) & 2(a_2a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix} \tag{12}$$

Of course, you could have expanded out the expression for R to show the equivalence. The least grungy way (in terms of messy algebra) was to show that $(I - \hat{a})R = I + \hat{a}$, where R is the expression for the rotation matrix in terms of a_1, a_2, and a_3 (Equation (13) below).

Part (c): There are multiple ways to solve this problem. The simplest way is to use the expression of part 5(b) (Equation (12) quoted in the text:

$$R = \frac{1}{1 + ||a||^2} \begin{bmatrix} 1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_3) & 2(a_1a_3 + a_2) \\ 2(a_1a_2 + a_3) & 1 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_1) \\ 2(a_1a_3 - a_2) & 2(a_2a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix} \tag{13}$$

where $||a||^2$ is shorthand notation for $||a||^2 = a_1^2 + a_2^2 + a_3^2$. Noting that

$$trace(R) = \frac{3 - ||a||^2}{1 + ||a||^2} \Rightarrow ||a||^2 = \frac{3 - trace(R)}{1 + trace(R)} = \frac{3 - r_{11} - r_{22} - r_{33}}{1 + r_{11} + r_{22} + r_{33}}$$

so that an expression for $||a||^2$ is known, simple algebraic manipulation of the off-diagonal term of R in Equation (13) yield

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1 + ||a||^2}{4} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Problem 6: Problem 10 (a,b) in Chapter 2 of MLS (not including the question of surjectivity in 10(b)).

Note that

$$\hat{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$\hat{\omega}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2 I; \quad \hat{\omega}^3 = -w^3 J$$

Hence the exponential of $\hat{\omega}$ can be computed as:

$$\begin{aligned}
\exp\left(\theta\hat{\omega}\right) &= \left(I + \frac{\theta}{1!}\hat{\omega} + + \frac{\theta^2}{2!}\hat{\omega}^2 + \cdots\right) \\
&= \left(I + \frac{w\theta}{1!}J - \frac{w^2\theta^2}{2!}I - \frac{w^3\theta^3}{3!}J + \cdots\right) \\
&= \left(1 - \frac{w^2\theta^2}{2!} + \cdots\right) I + \left(\frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \cdots\right) J \\
&= \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix}
\end{aligned}$$