

**ME 115(a): Solution to Homework #2**  
(Winter 2009/2010)

**Problem 1:** (15 points). To find the geometry of the moving centrode of the elliptical trammel, place a body fixed reference frame on the moving link so that its origin lies at the mid-point of Points **A** and **B**, and its  $x$ -axis point in the direction from point **A** to point **B**. In Figure 1(a) the basis vectors of this moving reference frame are denoted  $(\vec{x}_b, \vec{y}_b)$ . Let a fixed reference frame (with basis vectors  $(\vec{x}_f, \vec{y}_f)$ ) be placed at the intersection of the two sliding joints.

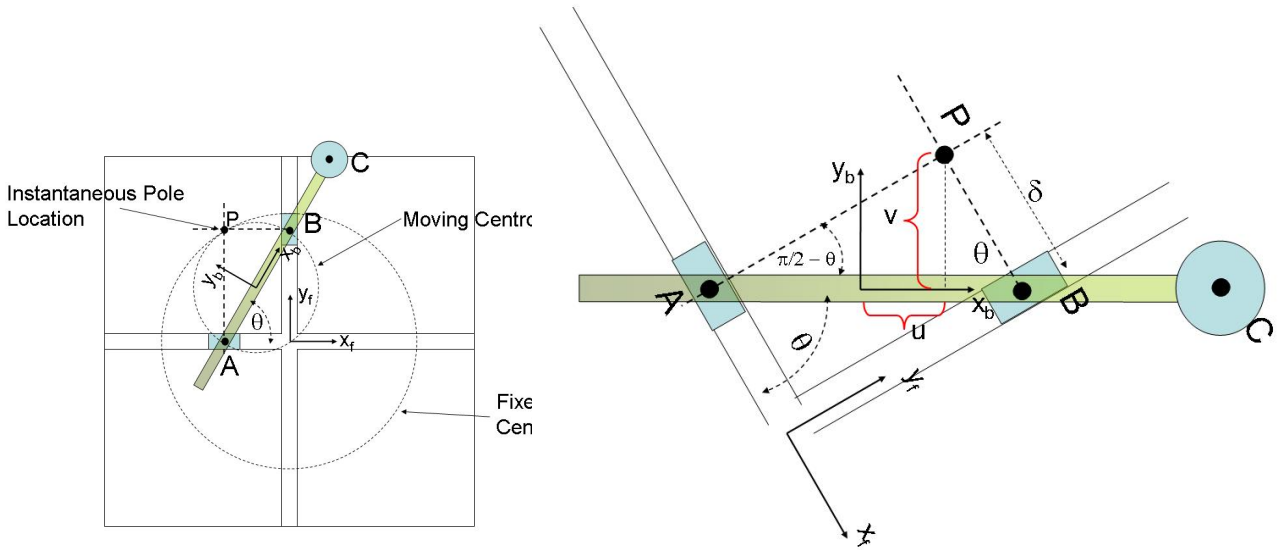


Figure 1: **(a):** Diagram of the Elliptical Trammel. **(b):** Expanded and rotated view of (a), showing the geometry of pole location in the moving coordinate system.

To solve this problem, one must compute the location of the centrode as seen by an observer in the moving frame. Let  $a = |\mathbf{AB}|$ . Let  $\theta$  denote the angle between the body-fixed  $x$ -axis and the  $x$ -axis of the fixed reference frame. This angle also defines the angles of the right-handed triangle **ABP**. Using the geometry of Figure 1(b), it can be seen that

$$\delta = |\mathbf{BP}| = a \sin \left( \frac{\pi}{2} - \theta \right) = a \cos \theta .$$

Similarly, from this diagram we can deduce that the  $x$ -coordinate of the centrode, denoted  $u$ , is given by:

$$u = \frac{a}{2} - \delta \cos \theta = \frac{a}{2} - a \cos^2 \theta .$$

Likewise, the  $y$ -coordinate of the centrode in the moving frame, denoted  $v$ , is simply:

$$v = \delta \sin \theta = a \cos \theta \sin \theta .$$

Thus, in the moving reference frame:

$$\begin{aligned}
u^2 + v^2 &= (a \cos \theta \sin \theta)^2 + \left(\frac{a}{2} - a \cos^2 \theta\right)^2 \\
&= a^2(\cos^2 \theta \sin^2 \theta + \frac{1}{4} + \cos^4 \theta - \cos^2 \theta) \\
&= a^2\left(\frac{1}{4} + \cos^2 \theta(\sin^2 \theta + \cos^2 \theta - 1)\right) \\
&= \left(\frac{a}{2}\right)^2
\end{aligned}$$

Thus, the moving centrode (the set of pole locations in the moving reference frame) is a circle with radius  $\frac{a}{2}$  centered at the midpoint of  $\vec{\mathbf{AB}}$ .

**Problem 2:** (15 points)

**Part (a):** You were to “prove” that a 3-dimensional body undergoing spherical motion has 3 degrees-of-freedom (DOF). By definition, a body undergoing spherical motion has one fixed point. Let the 3D body consist of  $N$  particles. Let  $P_1$  denote the particle lying at the fixed point. A point particle in 3-dimensional Euclidean space requires 3 independent variables to fix its location. However, since  $P_1$  does not move, it has 0 degrees-of-freedom (DOF). Now consider a particle  $P_2$  in the body. Particle  $P_2$  has 3 DOF as a particle. However, it is constrained to lie a fixed distance,  $d_{12}$  from particle  $P_1$  due to the fact that  $P_1$  and  $P_2$  are part of the same rigid body. The fixed distance relationship imposes one constraint on  $P_2$ . Next consider a point  $P_3$ , which lies a fixed distance from  $P_1$  and  $P_2$ . Therefore, there are two constraints on the location of  $P_3$ . Next, consider a particle  $P_4$ . Since it must lie a fixed distance from  $P_1$ ,  $P_2$ , and  $P_3$ , there are three constraints on its motion. Particles  $P_5, \dots, P_N$  similarly have 3 constraints.

Remembering that particle  $P_1$  has zero DOF, the total number of degrees of freedom of the  $N$  particles (considered independently) is:  $3(N - 1) + 0 = 3N - 3$ . The total number of constraints on these particles needed for them to form a rigid body is:  $1 + 2 + 3(N - 3) = 3N - 6$ . Hence, the total net DOF of a body is the number of freedoms of the particles minus the number of constraints that bind them into a rigid body:  $(3N - 3) - (3N - 6) = 3$ .

**Part (b):** (Problem 7 in MLS Chapter 2). There are several ways to solve this problem. One approach is analogous to the solution of Part (a). A rigid body in an  $n$ -dimensional Euclidean space consists of particles  $P_1, P_2, \dots, P_N$ . Each particle requires  $n$  independent DOF to uniquely describe its state.

- Arbitrarily pick particle  $P_1$  in the body. There are no constraints on its motion.
- Particle  $P_2$  is constrained by a single constraint to lie a fixed distance from  $P_1$ .
- Particle  $P_3$  is similarly constrained by its fixed distance from  $P_1$  and  $P_2$ .
- Continuing in this way, particle  $P_{n+1}$  is constrained to have  $n$  fixed distances to particles  $P_1, \dots, P_n$ .

As independent particles, the  $N$  bodies have a total of  $nN$  DOF. However, these bodies are constrained by  $N_C$  constraints, where<sup>1</sup>:

$$\begin{aligned} N_C &= 0 + 1 + 2 + \cdots + (n-1) + (N-n)n = \frac{1}{2}((n-1)^2 + (n-1)) + (N-n)n \\ &= \frac{1}{2}(n^2 - n) + Nn - n^2 = Nn - \frac{1}{2}(n^2 + n). \end{aligned} \quad (1)$$

Hence, the total number of DOF can be found as

$$Nn - \left( Nn - \frac{1}{2}(n^2 + n) \right) = \frac{1}{2}(n^2 + n).$$

Here is an alternative solution. The location of a body moving in an  $n$ -dimensional Euclidean space can be uniquely described by a point in the body,  $p \in \mathbb{R}^n$  and a rotation matrix,  $R \in SO(n)$ . Recall that matrices in  $SO(n)$  must satisfy the relation  $R^T R = I$ , where  $I$  is the  $n \times n$  identity matrix. Thus, for the entries of any matrix in  $SO(3)$ , there are  $\frac{1}{2}(n + n^2)$  constraints of the form:

$$c_i^T c_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

where  $c_i$  is the  $i^{th}$  column of  $R$ . Thus, the  $n^2$  elements of  $R \in SO(n)$  can only have  $n^2 - \frac{1}{2}(n + n^2) = \frac{1}{2}(n^2 - n)$  independent DOF. Thus, a rigid body in  $\mathbb{R}^n$  has

$$n + \frac{1}{2}(n^2 - n) = \frac{1}{2}(n^2 + n)$$

DOF.

**Problem 3:** (5 Points). Problem 3(c) in Chapter 2 of MLS.

Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = [\vec{r}_1 \quad \vec{r}_2 \quad \vec{r}_3].$$

Expanding  $\det(R)$  using cofactors, one finds:

$$\begin{aligned} \det(R) &= r_{11}(r_{22}r_{33} - r_{32}r_{23}) + r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{12}r_{23} - r_{22}r_{13}) \\ &= \vec{r}_1^T \cdot (\vec{r}_2 \times \vec{r}_3) \end{aligned}$$

**Problem 4:** (10 Points). Problem 4(a,b) in Chapter 2 of MLS.

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<sup>1</sup>This result is obtained by using the identity:

$$\sum_{j=1}^k j = \frac{1}{2}(k^2 + k).$$

**Part (a):** Let's assume that the statement in part (b) of the problem is true. Let  $\vec{w}$  be a  $3 \times 1$  vector and let  $\vec{v}$  be any  $3 \times 1$  vector. Then:

$$\begin{aligned}
 (R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\
 &= R(\vec{w} \times (R^T\vec{v})) \\
 &= (R\vec{w}) \times (RR^T\vec{v}) \\
 &= (R\vec{w}) \times \vec{v} \\
 &= (\widehat{R\vec{w}})\vec{v}
 \end{aligned}$$

Since this must be true for any vector  $\vec{v}$ , then  $R\hat{w}R^T = \widehat{R\vec{w}}$ .

**Part (b):** We can now assume that part (a) holds.

$$\begin{aligned}
 (R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\
 &= (R\hat{v}R^T)(R\vec{w}) \\
 &= R\hat{v}R^TR\vec{w} \\
 &= R(\hat{v}\vec{w}) \\
 &= R(\vec{v} \times \vec{w})
 \end{aligned}$$

**Problem 5:** (10 points). Problem 10 (b,c) in Chapter 2 of MLS.

**Problem 10(b):** Note that

$$\hat{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$\hat{\omega}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2 I; \quad \hat{\omega}^3 = -w^3 J; \quad \text{etc..}$$

Hence the exponential of  $\hat{\omega}$  can be computed as:

$$\begin{aligned}
 \exp(\theta\hat{\omega}) &= \left( I + \frac{\theta}{1!}\hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \dots \right) \\
 &= \left( I + \frac{w\theta}{1!}J - \frac{w^2\theta^2}{2!}I - \frac{w^3\theta^3}{3!}J + \dots \right) \\
 &= \left( 1 - \frac{w^2\theta^2}{2!} + \dots \right) I + \left( \frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \dots \right) J \\
 &= \cos(w\theta)I + \sin(w\theta)J = \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix}
 \end{aligned}$$

**Problem 10(c):** Let  $R \in SO(2)$  and  $\hat{\omega} \in so(2)$  have the forms:

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}. \quad (2)$$

A brute force calculation yields:

$$\begin{aligned}
R\hat{\omega}R^T &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \omega \sin \theta & -\omega \cos \theta \\ \omega \cos \theta & \omega \sin \theta \end{bmatrix} \\
&= \begin{bmatrix} \omega(\cos \theta \sin \theta - \sin \theta \cos \theta) & -\omega(\cos^2 \theta + \sin^2 \theta) \\ \omega(\sin^2 \theta + \cos^2 \theta) & -\omega(-\sin \theta \cos \theta + \sin \theta \cos \theta) \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}
\end{aligned} \tag{3}$$