

## ME 115(a): Solution to Homework #2

**Problem 1:** (10 points) Can every orthogonal matrix be represented by the exponential of a real matrix?

You should have either remembered or derived the fact that  $\det(e^C) = e^{\text{tr}(C)}$ , where  $\text{tr}(C)$  is the trace of determinant  $C$ . Note that if  $\text{tr}(C)$  is real, then  $e^{\text{tr}(C)}$  is always a positive number and therefore orthogonal matrices with determinant -1 can not be represented as a matrix exponential. This arises from the fact that the Orthogonal Group is a disconnected group. That is, the matrices with +1 determinant are all connected to each other, and similarly the ones with -1 determinant form a continuous subset of  $\mathcal{O}(3)$ . But, the two subsets are disjoint.

Let's consider this solution in a bit more detail. Note, that if  $\text{tr}(C) = \frac{\pi}{2}i$  (where  $i^2 = -1$ ), then  $\det(e^C) = -1$ . However, this can not be true if  $C$  is real. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Let  $C$  be a  $n \times n$  matrix. If  $n$  is even, then all of the eigenvalues of  $C$  must be complex conjugates and/or an even number of real roots. Thus, the sum of the eigenvalues must be real. Similarly, if  $n$  is odd, the eigenvalues will either be complex conjugates and/or an odd number of real eigenvalues. Thus, the sum of the eigenvalues must also be real number. Thus, if  $C$  is a real matrix, then  $e^C$  can not represent orthogonal matrices with determinant -1.

**Problem 2:** (Problem 4(a,b,c) in Chapter 2 of MLS).

**Part (a):** Let's assume that the statement in part (b) of the problem is true. Let  $\vec{w}$  be a  $3 \times 1$  vector and let  $\vec{v}$  be any  $3 \times 1$  vector. Then:

$$\begin{aligned} (R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\ &= R(\vec{w} \times (R^T\vec{v})) \\ &= (R\vec{w}) \times (RR^T\vec{v}) \\ &= (R\vec{w}) \times \vec{v} \\ &= \widehat{(R\vec{w})}\vec{v} \end{aligned}$$

Since this must be true for any vector  $\vec{v}$ , then  $R\hat{w}R^T = \widehat{(R\vec{w})}$ .

**Part (b):** We can now assume that part (a) holds.

$$\begin{aligned} (R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\ &= (R\hat{v}R^T)(R\vec{w}) \\ &= R\hat{v}R^TR\vec{w} \\ &= R(\hat{v}\vec{w}) \\ &= R(\vec{v} \times \vec{w}) \end{aligned}$$

**Part (c):** to show that  $so(3)$  is a vector space, note that all elements in  $so(3)$ , which consists of  $3 \times 3$  skew symmetric matrices, can be written as a linear combination of the following 3 elements of  $so(3)$  (which therefore form a basis for  $so(3)$ ):

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (1)$$

Moreover, the linearity requirements on vectors can be seen as follows, for  $\alpha, \beta \in \mathbb{R}$ ,  $\vec{v}, \vec{\omega} \in \mathbb{R}^3$ , and  $\hat{v}, \hat{\omega} \in so(3)$ :

- $\alpha \hat{v} \in so(3)$ .
- $(\alpha \vec{v} + \beta \vec{\omega})^\wedge = (\alpha \hat{v} + \beta \hat{\omega}) \in so(3)$ .

**Problem 3:** (5 points, Problem 5(c) of chapter 2 in the MLS text). There are two ways to solve this. The simplest way is to use the result of part 5(b) quoted in the text:

$$R = \frac{1}{1 + ||a||^2} \begin{bmatrix} 1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1 a_2 - a_3) & 2(a_1 a_3 + a_2) \\ 2(a_1 a_2 + a_3) & 1 - a_1^2 + a_2^2 - a_3^2 & 2(a_2 a_3 - a_1) \\ 2(a_1 a_3 - a_2) & 2(a_2 a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix} \quad (2)$$

where  $||a||^2$  is shorthand notation for  $||a||^2 = a_1^2 + a_2^2 + a_3^2$ . Noting that

$$trace(R) = \frac{3 - ||a||^2}{1 + ||a||^2} \Rightarrow ||a||^2 = \frac{3 - trace(R)}{1 + trace(R)} = \frac{3 - r_{11} - r_{22} - r_{33}}{1 + r_{11} + r_{22} + r_{33}}$$

so that an expression for  $||a||^2$  is known, simple algebraic manipulation of the off-diagonal term of  $R$  yield

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1 + ||a||^2}{4} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

If you didn't use the results of 5(b) in the text, then you would have started with Cayley's formula  $R = (I - \hat{a})^{-1}(I + \hat{a})$  and derived Equation (2).

**Problem 4:** (10 points, Problem 8(b,c) in Chapter 2 of the MLS text).

**Solution to 8(b):**

$$\begin{aligned} e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \dots \\ &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \dots \\ &= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \dots)g^{-1} \\ &= ge^{\Lambda}g^{-1} \end{aligned}$$

**Solution to 8(c):**

**Problem 5:** (5 points, Problem 10(b) in Chapter 2 of the MLS text).

Note that

$$\hat{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$\hat{\omega}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2 I; \quad \hat{\omega}^3 = -w^3 J$$

Hence the exponential of  $\hat{\omega}$  can be computed as:

$$\begin{aligned} \exp(\theta\hat{\omega}) &= \left( I + \frac{\theta}{1!}\hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \cdots \right) \\ &= \left( I + \frac{w\theta}{1!}J - \frac{w^2\theta^2}{2!}I - \frac{w^3\theta^3}{3!}J + \cdots \right) \\ &= \left( 1 - \frac{w^2\theta^2}{2!} + \cdots \right) I + \left( \frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \cdots \right) J \\ &= \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix} \end{aligned}$$

While you weren't asked to consider this part of the problem, note that the exponential map from  $so(2)$  to  $SO(3)$  can not be surjective, as every point in  $SO(3)$  can not be covered by every point in  $so(2)$ . This map is not injective since  $\exp(\theta\hat{\omega}) = \exp((\theta + 2\pi)\hat{\omega})$ .

**Problem 6:** (5 points). You were to find the axis and angle of rotation for the following rotation matrix:

$$\begin{bmatrix} 0.866025 & -0.353553 & 0.353553 \\ 0.353553 & 0.933013 & 0.0669873 \\ -0.353553 & 0.0669873 & 0.933013 \end{bmatrix}.$$

Let the matrix entries be denoted  $a_{ij}$ , where  $i = 1, 2, 3$  denotes the row, and  $j = 1, 2, 3$  denotes the column. From class notes or the text:

$$\cos(\phi) = \frac{a_{11} + a_{22} + a_{33} - 1}{2} = \frac{0.866025 + 0.933013 + 0.933013 - 1.0}{2} = 0.8660245$$

where  $\phi$  is the angle of rotation. Thus,  $\phi = \cos^{-1}(0.8660245) = 30^\circ$ . Thus,  $\sin(\phi) = 0.5$ , and

therefore:

$$s_x = \frac{a_{32} - a_{23}}{2 \sin \phi} = 0.0 \quad (3)$$

$$s_y = \frac{a_{13} - a_{31}}{2 \sin \phi} = 0.707106 \quad (4)$$

$$s_z = \frac{a_{21} - a_{12}}{2 \sin \phi} = 0.707106 \quad (5)$$