ME 115(b): Homework #2

Problem 1: (Special Configurations of "Slider-Crank" linkages).

Part (a): Let's establish a coordinate system with origin at joint 1, with the x-axis colinear with the piston axis, with the y-axis normal to the x-axis as shown, and with a z-axis point out of the plane of the mechanism, so that x-y-z axes form a right handed coordinate system (see Figure 1).

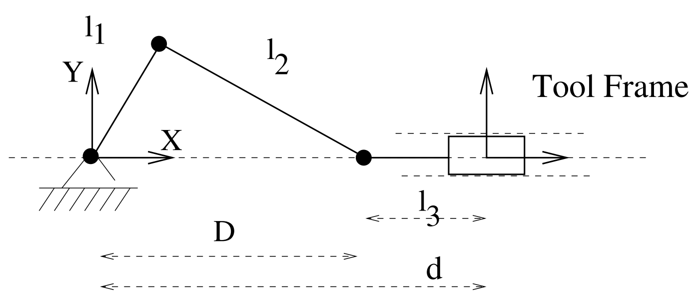

Figure 1: Slider Crank Mechanism

While the device is planar, for convenience let's express the twist coordinates for each joint in the full 3-D space. The twist coordinates for revolute joints #1 and #2 are straightforward:

$$\begin{aligned} \xi_1 &= [0\ 0\ 0\ 0\ 0\ 1]^T \\ \xi_2 &= [(l_1 s_1)\ \ -(l_1 c_1)\ 0\ 0\ 0\ 1]^T \end{aligned} \tag{1}$$

where s_1 and c_1 are respectively shorthand for $\sin\theta_1$ and $\cos\theta_1$, and the angle θ_1 is defined as the angle between the x-axis and the first link (the first link, whose length is labeled l_1 in Figure 1. Since joint #3 is always passes through the x-axis, its associated screw can be represented as:

$$\xi_3 = [0\ \ -D\ 0\ 0\ 0\ 1]^T \tag{2}$$

where the distance, D, is defined in Figure 1, and represents the distance between the first and third joint axes. From the diagram, it can be seen that D and takes the value:

$$D = l_1 c_1 + l_2 c_{12}\ . \tag{3}$$

where $c_{12} = \cos(\theta_1 + \theta_2)$, and θ_2 is defined as the angle between the extension of the first link and the line representing the second link.

The fourth joint is prismatic, and therefore an infinite pitch joint. Its screw coordinates are simply:

$$\xi_4 = [1\ 0\ 0\ 0\ 0\ 0]^T \tag{4}$$

Part (b):

A stationary configuration of joint 4 will occur when:

$$det[\begin{bmatrix} \xi_1 \cdot \xi_1 & \xi_1 \cdot \xi_2 & \xi_1 \cdot \xi_3 \\ \xi_2 \cdot \xi_1 & \xi_2 \cdot \xi_2 & \xi_2 \cdot \xi_3 \\ \xi_3 \cdot \xi_1 & \xi_3 \cdot \xi_2 & \xi_3 \cdot \xi_3 \end{bmatrix} = 0 \ . \tag{5}$$

For this linkage:

$$\begin{aligned} \xi_1 \cdot \xi_1 &= \xi_1 \cdot \xi_2 \ = \ \xi_1 \cdot \xi_3 \ = \ 1 \\ \xi_2 \cdot \xi_2 &= 1 + l_1^2 \\ \xi_2 \cdot \xi_3 &= 1 + l_1 c_1 D \\ \xi_3 \cdot \xi_3 &= 1 + D^2 \end{aligned} \quad . \tag{6}$$

Substituting Equation (6) into Equation (5) and taking the determinant:

$$det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 + l_1^2 & 1 + l_1 c_1 D \\ 1 & 1 + l_1 c_1 D & 1 + D^2 \end{bmatrix} = l_1^2 D^2 (1 - c_1^2) = l_1^2 D^2 s_1^2 \tag{7}$$

Clearly, Equation (7) is zero when $\theta_1 = 0$, or when $\theta_1 = \pm\pi$. Physically, this situation occurs when the first link lies along the x-axis. Also, Equation (7) will take a zero value when D $=0$. This latter situation can only occur when $l_1 = l_2$ and $\theta_2 = \pm\pi$ (so that joint 2 is rotated until links 1 and 2 are superimposed).

Part (c):

A stationary configuration of joint 1 will occur when:

$$det \begin{bmatrix} \xi_2 \cdot \xi_2 & \xi_2 \cdot \xi_3 & \xi_2 \cdot \xi_4 \\ \xi_3 \cdot \xi_2 & \xi_3 \cdot \xi_3 & \xi_3 \cdot \xi_4 \\ \xi_4 \cdot \xi_2 & \xi_4 \cdot \xi_3 & \xi_4 \cdot \xi_4 \end{bmatrix} = 0 \ . \tag{8}$$

In addition to the quantities derived in Part (b):

$$\begin{aligned} \xi_2 \cdot \xi_4 &= \xi_4 \cdot \xi_2 \ = \ l_1 s_1 \\ \xi_3 \cdot \xi_4 &= \xi_4 \cdot \xi_3 \ = \ 0 \\ \xi_4 \cdot \xi_4 &= 1 \end{aligned} \quad . \tag{9}$$

Consequently a stationary configuration will occur when

$$det \begin{bmatrix} 1 + l_1^2 & 1 + l_1 c_1 D & l_1 s_1 \\ 1 + l_1 c_1 D & 1 + D^2 & 0 \\ l_1 s_1 & 0 & 1 \end{bmatrix} = l_1^2 c_1^2 + D^2 - 2 l_1 c_1 D = (l_1 c_1 - D)^2 \ . \tag{10}$$

But, since $D = l_1 c_1 + l_2 c_{12}$, joint #1 will have a stationary configuration when:

$$l_2 \cos(\theta_1 + \theta_2) = 0 \ . \tag{11}$$

Equation 11 will be satisfied when $\theta_1 + \theta_2 = \pm\frac{\pi}{2}$, which can only happen when $l_1 \geq l_2$. Such a situation will occur in configurations where link 2 is vertically aligned. If $l_1 > l_2$, then joint #1 is unable to rotate completely through 360^o, and the stationary points are the limits of travel.

Problem 2: Problem 21, Chapter 3 of MLS.

Part (a): We can think of this mechanism as consisting of 8 bodies (noticing that each prismatic joint can be thought of as two bodies). The total number of *relative* degrees of freedom possed by this collection of bodies is 3(8-1) = 21. This collection of bodies is held together by 6 revolute joints and 3 prismatic joints. Thus, the number of constraints is $9 \cdot 2 = 18$. Therefore, the mechanism as 21-18=3 internal degrees of freedom.

Part (b): The structure equations can be derived either using the **P**roduct **O**f **E**xponentials (POE) method, or the Denavit-Hartenberg (DH) convention. Let's consider the POE approach. In this approach, the structure equations take the form:

$$e^{\xi_{11}\alpha_1} e^{\xi_{12}d_1} e^{\xi_{13}\beta_1} g_{bt}(0) = e^{\xi_{21}\alpha_2} e^{\xi_{22}d_2} e^{\xi_{23}\beta_2} g_{bt}(0) = e^{\xi_{31}\alpha_3} e^{\xi_{32}d_3} e^{\xi_{33}\beta_3} g_{bt}(0)$$

Using the obvious "home position," (the one showed in the left hand diagram of the figure which goes along with problem 3.21 in the MLS text) the twists are:

$$\begin{aligned}
\xi_{11} &= \begin{bmatrix} h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \xi_{13} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{21} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \xi_{23} = \begin{bmatrix} -h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{31} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{32} = \frac{1}{\sqrt{w^2+b^2}} \begin{bmatrix} w \\ h \\ 0 \end{bmatrix}; \quad \xi_{33} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix}
\end{aligned} \tag{12}$$

where α_j is the angle made between the axis of the j^{st} prismatic joint axis and the x-axis of the stationary, or base, frame. I.e., this is the angle made the first passive revolute joint in the j^{th} serial chain.

Note that the angles α_i and β_i can be found using the law of cosines. For example:

$$\cos\alpha_1 = \frac{h^2 + d_1^2 - d_3^2}{2hd_1}$$

Part (c): Using the notation in the figure above, we can see that the location of the points $P_1, \cdots, P_4$ are:

$$\begin{aligned} P_1 &= \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \qquad P_2 = \begin{bmatrix} 0 \\ -h/2 \end{bmatrix}; \\ P_3 &= \begin{bmatrix} x \\ y \end{bmatrix}; \qquad +R_\phi \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \qquad P_4 = \begin{bmatrix} x \\ y \end{bmatrix}; \qquad +R_\phi \begin{bmatrix} 0 \\ -h/2 \end{bmatrix}; \end{aligned} \tag{13}$$

The actuator lengths can be derived as:

$$\begin{aligned} d_1 &= ||P_3 - P_1|| \\ d_2 &= ||P_4 - P_2|| \\ d_3 &= ||P_3 - P_2|| \end{aligned} \tag{14}$$

Part (d):

The spatial Jacobians are simply:

$$\begin{bmatrix} \xi_{11} & \xi_{12}^{'} & \xi_{13}^{'} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{d}_1 \\ \dot{\beta}_1 \end{bmatrix} = \begin{bmatrix} \xi_{21} & \xi_{22}^{'} & \xi_{23}^{'} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_2 \\ \dot{d}_2 \\ \dot{\beta}_2 \end{bmatrix} = \begin{bmatrix} \xi_{31} & \xi_{32}^{'} & \xi_{33}^{'} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_3 \\ \dot{d}_3 \\ \dot{\beta}_3 \end{bmatrix}$$

where $\xi_{ij}^{'}$ is the appropriately transformed twist coordinates. In this case, the transformed twists can be determined by inspection:

$$\begin{aligned} \xi_{12}^{'} &= \begin{bmatrix} \cos\alpha_1 \\ \sin\alpha_1 \\ 0 \end{bmatrix}; \qquad \xi_{13}^{'} = \begin{bmatrix} (w+d_1)\sin\alpha_1 + h/2 \\ -(w+d_1)\cos\alpha_1 \\ 1 \end{bmatrix} \\ \xi_{22}^{'} &= \begin{bmatrix} w\cos\alpha_1 - h\sin\alpha_1 \\ w\sin\alpha_1 + h\cos\alpha_1 \\ 0 \end{bmatrix}; \qquad \xi_{23}^{'} = \begin{bmatrix} (w+d_2)\sin\alpha_2 - h/2 \\ -(w+d_2)\cos\alpha_2 \\ 1 \end{bmatrix} \\ \xi_{32}^{'} &= \begin{bmatrix} \cos\alpha_2 \\ \sin\alpha_2 \\ 0 \end{bmatrix}; \qquad \xi_{33}^{'} = \xi_{13}^{'} \end{aligned} \tag{15}$$

Part (e):

To solve for the singular configurations, we know that such configurations exist when the determinant of the hybrid Jacobian is equal to 0 (or more formally, when the hybrid Jacobian loses rank). So, we begin by forming the hybrid Jacobian:

$$
\begin{aligned}
J^h_{ST_1} &= \begin{bmatrix} \frac{\partial \overrightarrow{d^1_{ST}}}{\partial \alpha_1} & \frac{\partial \overrightarrow{d^1_{ST}}}{\partial d_1} & \frac{\partial \overrightarrow{d^1_{ST}}}{\partial \beta_1} \end{bmatrix} \\
J^h_{ST_2} &= \begin{bmatrix} \frac{\partial \overrightarrow{d^2_{ST}}}{\partial \alpha_2} & \frac{\partial \overrightarrow{d^2_{ST}}}{\partial d_2} & \frac{\partial \overrightarrow{d^2_{ST}}}{\partial \beta_2} \end{bmatrix} \\
J^h_{ST_3} &= \begin{bmatrix} \frac{\partial \overrightarrow{d^3_{ST}}}{\partial \alpha_3} & \frac{\partial \overrightarrow{d^3_{ST}}}{\partial d_3} & \frac{\partial \overrightarrow{d^3_{ST}}}{\partial \beta_3} \end{bmatrix}
\end{aligned} \tag{16}
$$

where each of the vectors, $\overrightarrow{d^i_{ST}}$, correspond to the position vectors of the resultant structure equations:

$$
\begin{aligned}
g^1_{ST} &= \begin{bmatrix} R & \overrightarrow{d^1_{ST}} \\ 0 & 1 \end{bmatrix} \\
g^2_{ST} &= \begin{bmatrix} R & \overrightarrow{d^2_{ST}} \\ 0 & 1 \end{bmatrix} \\
g^3_{ST} &= \begin{bmatrix} R & \overrightarrow{d^3_{ST}} \\ 0 & 1 \end{bmatrix}
\end{aligned} \tag{17}
$$

Using the structure equations solved earlier, we obtain the following position expressions:

$$
\begin{aligned}
\overrightarrow{d^1_{ST}} &= \begin{bmatrix} (d_1 + w)\cos\alpha_1 + \frac{h}{2}\sin(\alpha_1 + \beta 1) \\ -\frac{h}{2}(-1 + \cos(\alpha_1 + \beta_1)) + (d_1 + w)(\sin\alpha_1) \end{bmatrix} \\
\overrightarrow{d^2_{ST}} &= \begin{bmatrix} (d_2 + w)\cos\alpha_2 - \frac{h}{2}\sin(\alpha_2 + \beta 2) \\ \frac{h}{2}(-1 + \cos(\alpha_2 + \beta_2)) + (d_2 + w)(\sin\alpha_2) \end{bmatrix} \\
\overrightarrow{d^3_{ST}} &= \begin{bmatrix} (d_3 + w)\cos\alpha_3 + \frac{h}{2}\sin(\alpha_3 + \beta 3) \\ -\frac{h}{2}(-1 + \cos(\alpha_3 + \beta_3)) + (d_3 + w)(\sin\alpha_3) \end{bmatrix}
\end{aligned} \tag{18}
$$

Now solving for the hybrid Jacobians for each serial chain, we get:

$$
\begin{aligned}
J^h_{ST_1} &= \begin{bmatrix} \frac{h}{2}\cos(\alpha_1 + \beta_1) - (d_1 + w)\sin\alpha_1 & \cos\alpha_1 & \frac{h}{2}\cos(\alpha_1 + \beta_1) \\ (d_1 + w)\cos\alpha_1 + \frac{h}{2}\sin(\alpha_1 + \beta_1) & \sin\alpha_1 & \frac{h}{2}\sin(\alpha_1 + \beta_1) \end{bmatrix} \\
J^h_{ST_2} &= \begin{bmatrix} -\frac{h}{2}\cos(\alpha_2 + \beta_2) - (d_2 + w)\sin\alpha_2 & -\cos\alpha_2 & -\frac{h}{2}\cos(\alpha_2 + \beta_2) \\ (d_2 + w)\cos\alpha_2 - \frac{h}{2}\sin(\alpha_2 + \beta_2) & \sin\alpha_2 & -\frac{h}{2}\sin(\alpha_2 + \beta_2) \end{bmatrix} \\
J^h_{ST_3} &= \begin{bmatrix} \frac{h}{2}\cos(\alpha_3 + \beta_3) - (d_3 + w)\sin\alpha_3 & \cos\alpha_3 & \frac{h}{2}\cos(\alpha_3 + \beta_3) \\ (d_3 + w)\cos\alpha_3 + \frac{h}{2}\sin(\alpha_3 + \beta_3) & \sin\alpha_3 & \frac{h}{2}\sin(\alpha_3 + \beta_3) \end{bmatrix}
\end{aligned} \tag{19}
$$

Now solving for when the determinant loses rank:

$$
\begin{aligned}
det(J^1_{ST} * (J^1_{ST})^T) &= \tfrac{1}{8}\left[2h^2 + (8 + h^2)(d_1 + w)^2 + h^2(-2 + (d_1 + w)^2)\cos 2\beta_1 + 8h(d_1 + w)\sin\beta_1\right] \\
det(J^2_{ST} * (J^2_{ST})^T) &= \tfrac{1}{8}\left[2h^2 + (8 + h^2)(d_2 + w)^2 + h^2(-2 + (d_2 + w)^2)\cos 2\beta_2 - 8h(d_2 + w)\sin\beta_2\right] \\
det(J^3_{ST} * (J^3_{ST})^T) &= \tfrac{1}{8}\left[2h^2 + (8 + h^2)(d_3 + w)^2 + h^2(-2 + (d_3 + w)^2)\cos 2\beta_3 + 8h(d_3 + w)\sin\beta_3\right]
\end{aligned} \tag{20}
$$

Evaluation of the above equations indicate that singular configurations exist when:

- $d_1 + w = 0$ and $\beta_1 = 0, n\pi$

- $d_2 + w = 0$ and $\beta_2 = 0, n\pi$
- $d_3 + w = 0$ and $\beta_3 = 0, n\pi$

However, $d_2 + w = 0$ is not feasible due to the physical limitations of the actuator, requiring $d_2 + w > 0$. Therefore, the only realizable singular configurations are when $d_{1,3} + w = 0$ and $\beta_{1,3} = 0, n\pi$. These configurations correspond to either end of the object being flush against the base (or even both ends being flush).

With regards to the actuator singularities, we need to find under what conditions the span of the wrenches loses rank. We begin by solving for the wrenches of each serial chain. Recall the equation for a wrench is given by:

$$W_i = ||\vec{f_i}|| \begin{bmatrix} \vec{w}_i \\ \vec{\rho}_i \times \vec{w}_i \end{bmatrix} \tag{21}$$

Now applying the above expression to each chain, we get the following wrenches:

$$W_1 = ||\vec{f_1}|| \begin{bmatrix} \cos\beta_1 \\ -\sin\beta_1 \\ -\frac{h}{2}\cos\beta_1 \end{bmatrix} \tag{22}$$

$$W_2 = ||\vec{f_2}|| \begin{bmatrix} \sin\beta_2 \\ \cos\beta_2 \\ -\frac{h}{2}\sin\beta_2 \end{bmatrix} \tag{23}$$

$$W_3 = ||\vec{f_3}|| \begin{bmatrix} \cos\beta_3 \\ -\sin\beta_3 \\ \frac{h}{2}\cos\beta_3 \end{bmatrix} \tag{24}$$

Finally, taking the determinant of the span of the above wrenches, we get:

$$det \begin{bmatrix} \cos\beta_1 & \sin\beta_2 & \cos\beta_3 \\ -\sin\beta_1 & \cos\beta_2 & -\sin\beta_3 \\ -\frac{h}{2}\cos\beta_1 & -\frac{h}{2}\sin\beta_2 & \frac{h}{2}\cos\beta_3 \end{bmatrix} = h\cos(\beta_1 - \beta_2)\cos\beta_3 \tag{25}$$

Then setting the above expression equal to 0, we find the actuator singularities occur when $\beta_1 - \beta_2 = \frac{\pi}{2} + 2\pi k$ and/or similarly when $\beta_3 = \frac{\pi}{2} + 2\pi k$