ME 115(a): Solution to Homework #3

Problem # 1:

Part (a): Elements of $SU(2)$ have the form:

$$\begin{bmatrix} \mathbf{z} & \mathbf{w} \\ -\mathbf{w}^* & \mathbf{z}^* \end{bmatrix} = \begin{bmatrix} (a+ib) & (c+id) \\ -(c-id) & (a-ib) \end{bmatrix}$$

where $zz^* + ww^* = a^2 + b^2 + c^2 + d^2 = 1$. To show that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

form a basis for $SU(2)$, let A, B, C, and D be real numbers. Then, the matrix formed by the product of A, B, C,and D with these matrices is:

$$A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + B\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + C\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + D\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} A+iB & C+iD \\ C-iD & A-iB \end{bmatrix}$$

is a matrix in $SU(2)$ for any choice of A, B, C, and D where $A^2 + B^2 + C^2 + D^2 = 1$. Thus these four basis matrices for $SU(2)$ are in 1-to-1 correspondence with the 1, i, j, and k basis elements for the quaternions. Thus, the scalar elements A, B, C, and D are in one-to-one correspondence with the scalar elements of unit quaternions. That is, let a unit quaternion be represented by $q = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The correspondence is then:

$$\lambda_1 = A = Re(z) = \frac{z + z^*}{2} \tag{1}$$

$$\lambda_2 = B = Im(z) = \frac{i(z^* - z)}{2} \tag{2}$$

$$\lambda_3 = C = Re(w) = \frac{w + w^*}{2} \tag{3}$$

$$\lambda_4 = D = Im(w) = \frac{i(w^* - w)}{2} \tag{4}$$

Part (b): The unit quaternion elements are in one-to-one correspondence with the Euler parameters of a rotation: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\cos\frac{\phi}{2}, \omega_x \sin\frac{\phi}{2}, \omega_y \sin\frac{\phi}{2}, \omega_z \sin\frac{\phi}{2})$. ϕ is the rotation about an axis represented by a unit vector $\vec{\omega} = [\omega_x\ \omega_y\ \omega_z]^T$. A 2×2 complex matrix which represents an arbitrary rotation as a function of the z-y-x Euler angles can be developed as the product of 2×2 complex matrices which represent rotations about the z, y, and x axes. A rotation about the x-axis of amount γ has the 2×2 representation (since $\lambda_1 = \cos\frac{\gamma}{2}$, $\lambda_2 = \sin\frac{\gamma}{2}$, $\lambda_3 = \lambda_4 = 0$):

$$\begin{bmatrix} (\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2}) & 0 \\ 0 & (\cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2}) \end{bmatrix} = \begin{bmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{bmatrix}$$

Similarly, a rotation of amount ϕ about the y-axis can be represented as:

$$\begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$

while a rotation of amount ψ about the z-axis can be represented as:

$$\begin{bmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}$$

The product of these matrices yields the result.

Part (c):

$$\phi = 2\cos^{-1}(a) = 2\cos^{-1}(\frac{\mathbf{z}+\mathbf{z}^*}{2}) \tag{5}$$

$$\omega_x = \frac{b}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{z}-\mathbf{z}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{6}$$

$$\omega_y = \frac{c}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}+\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{7}$$

$$\omega_z = \frac{d}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}-\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{8}$$

Problem 2: Problem 6(d,e) in Chapter 2 of MLS.

Part (d):

(i) If $A_1, A_2 \in SO(3)$, then each of the 9 elements in the product matrix $A_1\ A_2$ requires 3 multiplications and 2 additions. Hence, the product $A_1\ A_2$ requires a total of 27 multiplications and 18 additions.

(ii) Let q_1 and q_2 be quaternions, with respective real and vector parts q_{1R}, q_{2R} and $\vec{q}_{1P}$, $\vec{q}_{2P}$. The real part of the quaternion product, $q_{1R}q_{2R} - \vec{q}_{1P} \cdot \vec{q}_{2P}$, requires 4 multiplications and 3 additions (where the subtraction is counted as an addition). The pure part, $\vec{q}_{3P} = q_{1R}\vec{q}_{2P} + q_{2R}\vec{q}_{1P} + \vec{q}_{1P} \times \vec{q}_{2P}$, can be evaluated in 12 multiplications and 9 additions. Thus, the quaternion product requires a total of 16 multiplications and 12 additions. It is therefore more efficient than the equivalent matrix multiplication.

(iii) The rotation of a vector by multiplication of a 3×3 rotation matrix times a 3×1 vector requires only 9 multiplications and 6 additions.

(iv) The number of multiplications and additions for the equivalent quaternion operation will depend upon the form which one uses for the quaternion vector rotation. Using the identity $1 = q_R^2 + q_P \cdot q_P$, it is possible to show that the vector part of $q\tilde{x}q^{-1}$ in part (b) above can be rearranged to the form:

$$\vec{x} + 2[q_P \times (q_P \times \vec{x}) + q_R(q_P \times \vec{x})]$$

Since $q_P \times \vec{x}$ need only be evaluated once, this takes only 18 multiplications and 12 additions. However, no matter what form one tries, the quaternion approach will always take more operations than the matrix/vector approach for vector rotation.

Part (e): Let a body rotate about a fixed axis $\vec{\omega}$ by an angle $\phi(t)$. Let the unit quaternion $Q(t)$ denote the orientation of this rotating body. This quaternion can be represented in the "vector" form

$$Q(t) \; = \; \left(\cos(\frac{\phi(t)}{2}), \vec{\omega}\sin(\frac{\phi(t)}{2})\right) \; . \tag{9}$$

Thus, the time rate of change of this quaternion is:

$$\dot{Q}(t) = \left(-\frac{\dot{\phi}(t)}{2}\sin(\frac{\phi(t)}{2}), \frac{\dot{\phi}(t)}{2}\vec{\omega}\cos(\frac{\phi(t)}{2})\right) \; . \tag{10}$$

Recall that if $Q_1 = (Q_{1,r}, \vec{Q}_{1,v})$ and $Q_2 = (Q_{2,r}, \vec{Q}_{2,v})$ (in the convenient "vector" notation for a quaternion), then

$$Q_1 \cdot Q_2 = (Q_{1,r}Q_{2,r} - \vec{Q}_{1,v} \cdot \vec{Q}_{2,v}, Q_{1,r}\vec{Q}_{2,v} + Q_{2,r}\vec{Q}_{1,v} + \vec{Q}_{1,v} \times \vec{Q}_{2,v}) \; . \tag{11}$$

Substituting the appropriate forms of Equations (9) and (10) into Equation (11) yields:

$$\dot{Q} \cdot Q^* \; = \; \left(-\frac{\dot{\phi}}{2}\sin(\frac{\phi}{2}), \frac{\dot{\phi}}{2}\vec{\omega}\cos(\frac{\phi}{2})\right)\left(\cos(\frac{\phi}{2}), -\vec{\omega}\sin(\frac{\phi}{2})\right) \tag{12}$$

$$\; = \; \left(0, \frac{\dot{\phi}}{2}\vec{\omega}\right) \tag{13}$$

Problem 4: Problem 11(a,b,e) in Chapter 2 of MLS.

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi' = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi' = (h\hat{\xi}' h^{-1})^\vee \tag{14}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.$$

Expanding Eq. (14) gives:

$$\xi = (h\hat{\xi}' h^{-1})^{\vee} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$.

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\vec{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T \vec{p} + R\vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \vec{v}^s = R\vec{v}^b - R\hat{\omega}^b R^T \vec{p} = R\vec{v}^b - \hat{\omega}^s \vec{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

Using this result:

$$\vec{v}^s = R\vec{v}^b - \hat{\omega}^s \vec{p} = R\vec{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \end{bmatrix} \begin{bmatrix} \vec{v}^b \\ \omega^b \end{bmatrix}$$

Therefore:

$$V^s = \begin{bmatrix} \vec{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix} V^b$$

Problem 4: Problem 14 in Chapter 2 of MLS.

Part (a): Let $g \in SE(3)$ denote a homogeneous transformation matrix:

$$g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Then:

$$g^{-1} = \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_{g^{-1}} = \begin{bmatrix} R^T & -(\widehat{R^T\vec{p}})R^T \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

where we have made use of the identity $(\widehat{R^T\vec{p}}) = R^T\hat{p}R$. Let's now compute $Ad_g Ad_{g^{-1}}$:

$$Ad_g Ad_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Hence, $Ad_{g^{-1}}$ must equal $(Ad_g)^{-1}$ since $Ad_g Ad_{g^{-1}} = I$.

Part (b): If

$$g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Then

$$g_1 g_2 = \begin{bmatrix} R_1 R_2 & \vec{p}_1 + R_1\vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence:

$$\begin{aligned} Ad_{g_1 g_2} &= \begin{bmatrix} R_1R_2 & (\vec{p}_1 + R_1\vec{p}_2)\hat{} R_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_1^TR_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 & \hat{p}_1R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2R_2 \\ 0 & R_2 \end{bmatrix} = Ad_{g_1}Ad_{g_2} \end{aligned}$$

Problem 5: (Problem 18(a,b,c,d) in Chapter 2 of MLS).

Part (a): If a frame B is moving with respect to an observing frame A, then

$$\vec{V}^h_{AB} = \begin{bmatrix} \dot{\vec{p}}_{AB} \\ \vec{\omega}^s_{AB} \end{bmatrix} .$$

Note that

$$\vec{V}^h_{AB} = \begin{bmatrix} \dot{\vec{p}}_{AB} \\ \vec{\omega}^s_{AB} \end{bmatrix} = \begin{bmatrix} R_{AB}R^T_{AB}\dot{p}_{AB} \\ R_{AB}\vec{\omega}^b_{AB} \end{bmatrix} = \begin{bmatrix} R_{AB} & 0 \\ 0 & R_{AB} \end{bmatrix} \begin{bmatrix} R^T_{AB}\dot{\vec{p}}_{AB} \\ \vec{\omega}^b_{AB} \end{bmatrix} = \begin{bmatrix} R_{AB} & 0 \\ 0 & R_{AB} \end{bmatrix} \vec{V}^b_{AB}$$

Part (b): There are many ways to solve this problem. For example, you could either start with Proposition 2.14 or Proposition 2.15 on page 59 of MLS which relate the velocities of three frames, A, B, and C. Let's choose Prop. 2.15:

$$V^b_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc} \tag{15}$$

Using the fact that

$$V^h_{ac} = \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V^b_{ac}$$

Eq. (15) can be written as:

$$\begin{aligned} V_{ac}^h &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} (Ad_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b) \\ &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} \begin{bmatrix} R_{bc}^T & -R_{bc}^T \hat{p}_{bc} \\ 0 & R_{bc}^T \end{bmatrix} V_{ab}^b + \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V_{bc}^b \\ &= \begin{bmatrix} R_{ab} & -R_{ab}\hat{p}_{bc} \\ 0 & R_{ab} \end{bmatrix} V_{ab}^b + \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{bc} & 0 \\ 0 & R_{bc} \end{bmatrix} V_{bc}^b \\ &= \begin{bmatrix} I & -(\widehat{R_{ab}p_{bc}}) \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} V_{ab}^b + Ad_{R_{ab}} V_{bc}^h \\ &= Ad_{-R_{ab}p_{bc}} V_{ab}^h + Ad_{R_{ab}} V_{bc}^h \end{aligned} \tag{16}$$

Part (c): Let frames A and B be stationary "spatial" frames, and let Frame C be fixed to a moving body. Let V_{bc}^h be the hybrid velocity of the body, as seen by an observer in the B frame. If we now want to express this velocity as seem by an observer in the A frame (i.e., changing the spatial frame), we need to calculate V_{ac}^h. You can do this using the results of part (b) of this problem which derived the result:

$$V_{ac}^h = Ad_{-R_{ab}p_{bc}} V_{ab}^h + Ad_{R_{ab}} V_{bc}^h \tag{17}$$

If you chose this approach, then since A and B are stationary, $V_{ab}^h = 0$. Hence, Eq. (16) takes the form:

$$V_{ac}^h = Ad_{R_{ab}} V_{bc}^h$$

Hence, the hybrid velocity is dependent on the orientation of the spatial frame, but not its position.

Alternatively, if you don't want to rely upon part (b), you can recall that the expression for the hybrid velocity is:

$$V_{ac}^h = \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac}^s \end{bmatrix}$$

Since $\vec{p}_{ac} = \vec{p}_{ab} + R_{ab}\vec{p}_{bc}$, and $\vec{p}_{ab}$ is constant:

$$\dot{\vec{p}}_{ac} = R_{ab}\dot{\vec{p}}_{bc}.$$

Similarly, $\vec{\omega}_{ac} = R_{ab}\vec{\omega}_{bc}$. Hence, V_{ac}^h is dependent of $\vec{p}_{ab}$, but not R_{ab}.

Part (d): Let A be a stationary spatial frame. Let B and C be two different frames attached to a moving body. Let us assume that the velocity of the rigid body is given by V_{ab}^h. If we now switch the location of the body fixed frame from position B to position C, the hybrid velocity of the body is given by V_{ac}^h. Since B and C are both fixed in the body, then $V_{bc}^h = 0$ in Eq. (16). Hence Eq. (16) reduces to:

$$V_{ac}^h = Ad_{-R_{ab}p_{bc}} V_{ab}^h$$

Hence, the hybrid velocity in only dependent on p_{bc}, the position of the body frame, and not on R_{bc}, the orientation of the body fixed frame. Alternatively, you could compute V_{ac}^h in a "brute force" way:

$$\begin{aligned} V_{ac}^h &= \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}(\vec{p}_{ab} + R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ac}R_{ac}^T)^\vee \end{bmatrix} = \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}_{ab}^s R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ab}R_{bc}R_{bc}^T R_{ab}^T)^\vee \end{bmatrix} \\ &= \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}_{ab}^s R_{ab}\vec{p}_{bc}) \\ \vec{\omega}_{ab}^s \end{bmatrix} = Ad_{-R_{ab}p_{bc}} V_{ab}^h \end{aligned}$$

Thus, the result only depends upon $\vec{p}_{bc}$, and not R_{bc}.