ME 115(b): Homework #3 Solution
(Spring 2009/2010)

Problem 1: Problem 21 (a,b,c,d,e), Chapter 3 of MLS.

Part (a): We can think of this mechanism as consisting of 8 bodies (noticing that each prismatic joint can be thought of as two bodies). The total number of *relative* degrees of freedom possessed by this collection of bodies is 3(8-1) = 21. This collection of bodies is held together by 6 revolute joints and 3 prismatic joints. Thus, the number of constraints is $9 \cdot 2 = 18$. Therefore, the mechanism as 21-18=3 internal degrees of freedom.

Part (b): The structure equations can be derived either using the **P**roduct **O**f **E**xponentials (POE) method, or the Denavit-Hartenberg (DH) convention. Let's consider the POE approach. In this approach, the structure equations take the form:

$$e^{\xi_{11}\theta_{11}}e^{\xi_{12}(d_1-W)}e^{\xi_{13}\theta_{13}}g_{BT}(0) = e^{\xi_{21}\theta_{21}}e^{\xi_{22}(d_2-W)}e^{\xi_{23}\theta_{23}}g_{BT}(0) = e^{\xi_{31}\theta_{31}}e^{\xi_{32}(d_3-d_{3,0})}e^{\xi_{33}\theta_{33}}g_{BT}(0)$$

where $d_{3,0}$ is the length of the third prismatic joint in the home position, which is $d_{3,0} = \sqrt{h^2+W^2}$. With this home configuration,

$$g_{BT}(0) = \begin{bmatrix} I & \begin{bmatrix} W \\ 0 \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix}.$$

In the home position (the one showed in the left hand diagram of the figure which goes along with problem 3.21 in the MLS text) the twists are:

$$\begin{aligned}
\xi_{11} &= \begin{bmatrix} h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \xi_{13} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{21} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \xi_{23} = \begin{bmatrix} -h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{31} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \quad \xi_{32} = \frac{1}{\sqrt{w^2+b^2}}\begin{bmatrix} w \\ h \\ 0 \end{bmatrix}; \quad \xi_{33} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix}
\end{aligned} \tag{1}$$

Part (c): Let P_1, P_2, P_3, and P_4 denote the points in the plane where the revolute joint axes intersect the plane of the mechanism. For a given location (x,y) of the tool frame, and a given tool frame orientation, $R(\phi)$, the location of the points $P_1, \cdots, P_4$ are:

$$P_1 = \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \quad P_2 = \begin{bmatrix} 0 \\ -h/2 \end{bmatrix}; \quad P_3 = \begin{bmatrix} x \\ y \end{bmatrix} + R_\phi \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \quad P_4 = \begin{bmatrix} x \\ y \end{bmatrix} + R_\phi \begin{bmatrix} 0 \\ -h/2 \end{bmatrix}. \tag{2}$$

Using these formuli, the actuator lengths (i.e., the inverse kinematic solution) can be calculated as:

$$\begin{aligned}
d_1 &= ||P_3 - P_1|| - W = \sqrt{(x - \tfrac{h}{2}\sin\phi)^2 + (y + \tfrac{h}{2}(\cos\phi - 1))^2} - W \\
d_2 &= ||P_4 - P_2|| - W = \sqrt{(x + \tfrac{h}{2}\sin\phi)^2 + (y - \tfrac{h}{2}(\cos\phi - 1))^2} - W \\
d_3 &= ||P_3 - P_2|| - d_{3,0} = \sqrt{(x - \tfrac{h}{2}\sin\phi)^2 + (y - \tfrac{h}{2}(\cos\phi - 1))^2} - d_{3,0}
\end{aligned} \tag{3}$$

Part (d): The spatial Jacobians are simply:

$$J^s_{BT,1}\dot{\vec{\theta}}_1 = J^s_{BT,2}\dot{\vec{\theta}}_2 = J^s_{BT,3}\dot{\vec{\theta}}_3,$$

or

$$\begin{bmatrix}\xi_{11} & \xi'_{12} & \xi'_{13}\end{bmatrix}\begin{bmatrix}\dot{\theta}_{11}\\ \dot{d}_1\\ \dot{\theta}_{13}\end{bmatrix} = \begin{bmatrix}\xi_{21} & \xi'_{22} & \xi'_{23}\end{bmatrix}\begin{bmatrix}\dot{\theta}_{21}\\ \dot{d}_2\\ \dot{\theta}_{23}\end{bmatrix} = \begin{bmatrix}\xi_{31} & \xi'_{32} & \xi'_{33}\end{bmatrix}\begin{bmatrix}\dot{\theta}_{31}\\ \dot{d}_3\\ \dot{\theta}_{33}\end{bmatrix}$$

where ξ'_{ij} is the appropriately transformed twist coordinates. In this case, the transformed twists can be determined by inspection:

$$\begin{aligned}
\xi_{11} &= \begin{bmatrix}h/2\\0\\1\end{bmatrix}; \quad \xi'_{12} = \begin{bmatrix}\cos\theta_{11}\\ \sin\theta_{11}\\ 0\end{bmatrix}; \quad \xi'_{13} = \begin{bmatrix}(w + d_1)\sin\theta_{11} + h/2\\ -(w + d_1)\cos\theta_{11}\\ 1\end{bmatrix}\\
\xi_{21} &= \begin{bmatrix}-h/2\\0\\1\end{bmatrix}; \quad \xi'_{22} = \begin{bmatrix}\cos\theta_{21}\\ \sin\theta_{21}\\ 0\end{bmatrix}; \quad \xi'_{23} = \begin{bmatrix}(w + d_2)\sin\theta_{21} - h/2\\ -(w + d_2)\cos\theta_{21}\\ 1\end{bmatrix}\\
\xi_{31} &= \begin{bmatrix}-h/2\\0\\1\end{bmatrix}; \quad \xi'_{32} = \begin{bmatrix}\cos(\theta_{32} + \theta^o_{32})\\ \sin(\theta_{32} + \theta^o_{32})\\ 0\end{bmatrix}; \quad \xi'_{33} = \xi'_{13}
\end{aligned} \tag{4}$$

where θ^0_{32} is the angle made the d_3 with respect to the horizontal in the home position: $\theta^o_{32} = \tan^{-1}(\frac{h}{W})$.

Part (e): The set of possible motions of the tool frame lies in the intersection of the range spaces of the three Jacobians, $J^s_{BT,1}$, $J^s_{BT,2}$, $J^s_{BT,3}$. Thus, when *any* one of these Jacobians loses rank, the mechanism must be in an actuator singularity.

Taking the determinants of the above equations, singular configurations exist when:

- $d_1 + W = 0$,.
- $d_2 + W = 0$,
- $d_3 + d_{3,0} = 0$.

With regards to the actuator singularities, we need to find under what conditions the span of the wrenches generated by the actuators loses rank. We begin by solving for the wrenches of each serial chain. Recall the equation for a wrench is given by:

$$W_i = ||\vec{f}_i|| \begin{bmatrix} \vec{w}_i \\ \vec{\rho}_i \times \vec{w}_i \end{bmatrix} \tag{5}$$

Now applying the above expression to each chain, we get the following wrenches:

$$W_1 = ||\vec{f}_1|| \begin{bmatrix} \cos\theta_{11} \\ \sin\theta_{11} \\ \frac{h}{2}\cos\theta_{11} \end{bmatrix}; \qquad W_2 = ||\vec{f}_2|| \begin{bmatrix} \cos\theta_{21} \\ \sin\theta_{21} \\ -\frac{h}{2}\cos\theta_{21} \end{bmatrix}; \qquad W_3 = ||\vec{f}_3|| \begin{bmatrix} \cos\theta_{31}' \\ \sin\theta_{31}' \\ -\frac{h}{2}\cos\theta_{31}' \end{bmatrix} \tag{6}$$

where $\theta_{31}' = \theta_{31} + \theta_{31}^o$. Note that when $\theta_{21} = \theta_{31}'$, the third wrench becomes linearly dependent upon the second wrench, and therefore the span of actuator wrenches loses rank.

Problem 2:

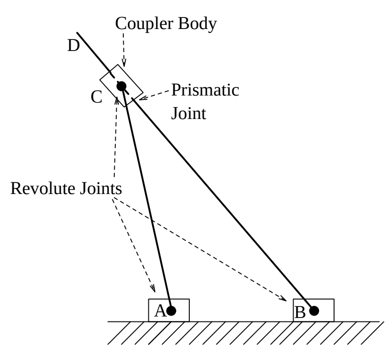

Figure 1: Mechanism

Part (a): The mobility is 1. There are 4 planar bodies coupled by 4 1-DOF joints, and therefore:

$$M = -3 + \sum_{i=1}^{4} f_i = -3 + 4 = 1.$$

Part (b): Assign a stationary reference frame with origin at point B in Figure 1, and with y-axis pointing vertical, x-axis horizontal to the right, and z-axis point out of the plane.. Let's consider the twist coordinates for joint axis A in this reference frame:

$$\xi_A = \begin{bmatrix} h_A\vec{\omega}_A + \vec{\rho}_a \times \vec{\omega}_A \\ \vec{\omega}_A \end{bmatrix} \tag{7}$$

where h_A is the twist of joint A, which is zero since A is a revolute joint. $\vec{\omega}_A$ is the unit vector collinear with the positive direction of joint axis A, which is simply the unit basis vector $\vec{z}$. The vector $\vec{\rho}_A$ points from the origin of the reference frame to any point on axis A–in this case it is $-|AB|\vec{x} = -\vec{x}$. Thus,

$$\xi_A = \begin{bmatrix} \vec{y} \\ \vec{z} \end{bmatrix} . \tag{8}$$

Similarly,

$$\xi_B = \begin{bmatrix} \vec{0} \\ \vec{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_C = \begin{bmatrix} d\sin\beta \\ d\cos\beta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_D = \begin{bmatrix} -\cos\beta \\ \sin\beta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

where β is the angle between $\vec{BA}$ and $\vec{BC}$ and $d = |BC|$. The dimension d can be determined from the angle β and the length $|AC|$ if needed.

Part (c): A stationary configuration of joint B will occur when the determinant of the cofactor of the (B, B) term in the Grammian matrix, G, is zero:

$$det(G) = det \begin{bmatrix} \xi_A \cdot \xi_A & \xi_A \cdot \xi_C & \xi_A \cdot \xi_D \\ \xi_C \cdot \xi_A & \xi_C \cdot \xi_C & \xi_C \cdot \xi_D \\ \xi_D \cdot \xi_A & \xi_D \cdot \xi_C & \xi_D \cdot \xi_D \end{bmatrix} = 0$$

Substituting the twist formulas from **Part (b)** into this formula yields:

$$det(G) \;=\; (d - \cos\beta)^2 \;=\; 0. \tag{10}$$

This equation will be satisfied under two conditions:

bf i. $d = \cos\beta$. This condition will occur when the line $\vec{AC}$ intersects the line $\vec{BC}$ orthogonally.

ii. $d = 0$ and $\cos\beta = 0$. In this case, points A, B, C, and D lie along a staight line.

For condition (i) to occur, it must be true that $|AC| < 1$. Condition (ii) can occur for any link sizes.

Part (d): The prismatic joint D will also have stationary configurations when all of the joint axes lie along a line.