ME 115(a): Solution to Homework #4

Problem 1:

Let the three rotating frames be termed the "ψ-frame," "ϕ-frame," and the "γ-frame." The spatial angular velocity of the body will be the same as the spatial angular velocity of the γ-frame:

$$\vec{\omega}^s = {}^S R^{\psi} \vec{\omega}^b_{\psi} + {}^S R^{\phi} \vec{\omega}^b_{\phi} + {}^S R^{\gamma} \vec{\omega}^b_{\gamma} \tag{1}$$

where $\vec{\omega}^b_{\psi}$ is the body angular velocity of the ψ-frame, ${}^S R^{\psi}$ is the orientation of the ψ-frame with respect to the stationary frame, etc. The body angular velocities are simply:

$$\vec{\omega}^b_{\psi} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \qquad \vec{\omega}^b_{\phi} = \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} \qquad \vec{\omega}^b_{\gamma} = \begin{bmatrix} \dot{\gamma} \\ 0 \\ 0 \end{bmatrix}$$

while the orientations of the frames are:

$${}^S R^{\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{\psi} R^{\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$${}^{\phi} R^{\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Using these frames, one can determine:

$${}^S R^{\phi} = {}^S R^{\psi\psi} R^{\phi} \qquad {}^S R^{\gamma} = {}^S R^{\psi\psi} R^{\phi\phi} R^{\gamma}.$$

Substituting into Equation (1) results in:

$$\vec{\omega}^s = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} -\dot{\phi}\sin\psi \\ \dot{\phi}\cos\psi \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\gamma}\cos\psi\sin\phi \\ \dot{\gamma}\sin\psi\sin\phi \\ \dot{\gamma}\cos\phi \end{bmatrix} = \begin{bmatrix} -\dot{\phi}\sin\psi + \dot{\gamma}\cos\psi\cos\phi \\ \dot{\phi}\cos\gamma + \dot{\gamma}\sin\psi\cos\phi \\ \dot{\psi} - \dot{\gamma}\sin\phi \end{bmatrix} \tag{2}$$

Problem 2: (Problem 11(d,e) in Chapter 2 of MLS). **Part (d):** Let

$$g = \begin{bmatrix} A & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix}$$

where $A \in SO(2)$ and $\vec{p} \in \mathbb{R}^2$. Then direct calculation shows that $\dot{g}g^{-1}$ and $g^{-1}\dot{g}$ are twists. The spatial and body velocities have definitions analogous to those for 3-dimensional rigid bodies

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\vec{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T \vec{p} + R\vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \vec{v}^s = R\vec{v}^b - R\hat{\omega}^b R^T \vec{p} = R\vec{v}^b - \hat{\omega}^s \vec{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

Using this result:

$$\vec{v}^s = R\vec{v}^b - \hat{\omega}^s \vec{p} = R\vec{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \tag{3}$$

Therefore:

$$V^s = \begin{bmatrix} \vec{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix} V^b$$

Problem 3: (Problem 13(a,b) in Chapter 2 of MLS).

Let $\xi_a = (-\vec{\omega}_a \times q_a + h\vec{\omega}_a, \vec{\omega}_a)$.

Part (a): The configuration of A relative to B is given by g_{ab}^{-1}:

$$g_{ab}^{-1} = \begin{bmatrix} R_{ab}^T & -R_{ab}^T \vec{p}_{ab} \\ \vec{0}^T & 1 \end{bmatrix}$$

Thus, the representation of $\vec{q}_a$ and $\vec{\omega}_a$ in B is:

$$\begin{aligned} \vec{q}_b &= R_{ab}^T \vec{q}_a - R_{ab}^T \vec{p}_{ab} \\ \vec{\omega}_b &= R_{ab}^T \vec{\omega}_a \end{aligned} \tag{4}$$

Substituting these equations into the expression:

$$\begin{aligned}
\xi_a &= \begin{bmatrix} (\vec{\omega}_b \times \vec{q}_b + h\vec{\omega}_b \\ \vec{\omega}_b \end{bmatrix} \\
&= \begin{bmatrix} -R_{ab}^T \hat{\omega}_a \vec{q}_a - R_{ab}^T \hat{p}_{ab} \vec{\omega}_a + hR_{ab}^T \vec{\omega}_a \\ R_{ab}^T \vec{\omega}_a \end{bmatrix} \\
&= \begin{bmatrix} R_{ab}^T & -R_{ab}^T \hat{p}_{ab} \\ 0 & R_{ab}^T \end{bmatrix} \begin{bmatrix} (\vec{\omega}_a \times \vec{q}_a + h\vec{\omega}_a \\ \vec{\omega}_a \end{bmatrix} \\
&= Ad_{g_{ab}^{-1}} \xi_z \qquad (5)
\end{aligned}$$

Part (b):

$$\begin{aligned}
\vec{q}' &= \vec{p} + R\vec{q}_a \\
\vec{\omega}' &= R\vec{\omega}_a
\end{aligned}$$

Hence,

$$\begin{aligned}
\xi' &= \begin{bmatrix} -\vec{\omega}' \times \vec{q}' + h\vec{\omega}' \\ \vec{\omega}' \end{bmatrix} = \begin{bmatrix} -R\vec{\omega}_a \times (p + Rq_a) + hR\vec{\omega}_a \\ R\vec{\omega}_a \end{bmatrix} \\
&= \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} -\vec{\omega} \times q_a + h\vec{\omega}_a \\ \vec{\omega}_a \end{bmatrix} \\
&= Ad_g \xi \qquad (6)
\end{aligned}$$

Problem 4: (Problem 14(b) in Chapter 2 of MLS:)
If

$$g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Then

$$g_1 g_2 = \begin{bmatrix} R_1 R_2 & \vec{p}_1 + R_1 \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence:

$$\begin{aligned}
Ad_{g_1 g_2} &= \begin{bmatrix} R_1 R_2 & (\vec{p}_1 + R_1 \vec{p}_2)\hat{} R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\
&= \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_1^T R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\
&= \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \\
&= \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2 R_2 \\ 0 & R_2 \end{bmatrix} = Ad_{g_1} Ad_{g_2}
\end{aligned}$$

Problem 5: (Problem 16(a,b) in Chapter 2 of MLS).

Part (a): $g_{0,3}$ can be determined in a variety of ways, such as by using the Denavit-Hartenberg, the product of exponentials (POE) approach, or a "brute force" approach.

Let's use the POE. Assume that the reference configuration is that given in Figure 2.17 of MLS. Hence, $g_{ST}(0)$ is:

$$g_{ST}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (l_1+l_2) \\ 0 & 0 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The twist coordinates of the joint axes (in the reference configuration) are:

$$\vec{\xi}_1 = \begin{bmatrix} h_1\vec{\omega}_1 + \rho_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{\xi}_2 = \begin{bmatrix} h_1\vec{\omega}_2 + \rho_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The forward kinematics is then given by

$$\begin{aligned} g_{ST} &= e^{\theta_1\hat{\xi}_1} e^{\theta_2\hat{\xi}_2} g_{ST}(0) \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & -(l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2)) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{7}$$

Part (b): Given g_{ST}, the spatial velocity can easily be computed as

$$\vec{V}^s_{ST} = (\dot{g}_{ST} g^{-1}_{ST})^\vee. \tag{8}$$

Later will will learn that one can formally rearrange these equations into the form:

$$\vec{V}^s_{ST} = J^s_{ST}\dot{\vec{\theta}}$$

where J^s_{ST} is termed the *spatial Jacobian matrix.* One could substitute Eq. (7) directly into Eq. (8) and carry through with the tedious algebra. To get a "hint" about the Jacobian matrix, note that

$$\begin{aligned} \dot{g}_{ST} g^{-1}_{ST} &= \frac{d}{dt}\left(e^{\theta_1\hat{\xi}_1} e^{\theta_1\hat{\xi}_2} g_{ST}(0)\right)\left(e^{\theta_1\hat{\xi}_1} e^{\theta_2\hat{\xi}_2} g_{ST}(0)\right)^{-1} \\ &= \left(\dot{\theta}_1\hat{\xi}_1 e^{\theta_1\hat{\xi}_1} e^{\theta_1\hat{\xi}_2} g_{ST}(0) + e^{\theta_1\hat{\xi}_1}\dot{\theta}_2\hat{\xi}_2 e^{\theta_2\hat{\xi}_2} g_{ST}(0)\right) g^{-1}_{ST}(0) e^{-\theta_2\hat{\xi}_2} e^{-\theta_1\hat{\xi}_1} \\ &= \dot{\theta}_1\hat{\xi}_1 + \dot{\theta}_2 e^{\theta_1\hat{\xi}_1}\hat{\xi}_2 e^{-\theta_1\hat{\xi}_1} \end{aligned} \tag{9}$$

Hence, the spatial Jacobian matrix takes the form:

$$\begin{aligned} J^s_{ST} &= \begin{bmatrix} \vec{\xi}_1 & \vec{\xi}_2' \end{bmatrix} = \begin{bmatrix} \vec{\xi}_1 & Ad_{e^{\theta_1 \hat{\xi}_1}} \vec{\xi}_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & l_1 \cos\theta_1 \\ 0 & l_1 \sin\theta_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Problem 6: (Problem 18(b,c,d) in Chapter 2 of MLS). **Part (b):** There are many ways to solve this problem. For example, you could either start with Proposition 2.14 or Proposition 2.15 on page 59 of MLS which relate the velocities of three frames, A, B, and C. Let's choose Prop. 2.15:

$$V^b_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc} \tag{10}$$

Using the fact that

$$V^h_{ac} = \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V^b_{ac}$$

Eq. (10) can be written as:

$$\begin{aligned} V^h_{ac} &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} (Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc}) \\ &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} \begin{bmatrix} R^T_{bc} & -R^T_{bc}\hat{p}_{bc} \\ 0 & R^T_{bc} \end{bmatrix} V^b_{ab} + \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V^b_{bc} \\ &= \begin{bmatrix} R_{ab} & -R_{ab}\hat{p}_{bc} \\ 0 & R_{ab} \end{bmatrix} V^b_{ab} + \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{bc} & 0 \\ 0 & R_{bc} \end{bmatrix} V^b_{bc} \\ &= \begin{bmatrix} I & -(\widehat{R_{ab}p_{bc}}) \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} V^b_{ab} + Ad_{R_{ab}} V^h_{bc} \\ &= Ad_{-R_{ab}p_{bc}} V^h_{ab} + Ad_{R_{ab}} V^h_{bc} \end{aligned} \tag{11}$$

Part (c): Let frames A and B be stationary "spatial" frames, and let Frame C be fixed to a moving body. Let V^h_{bc} be the hybrid velocity of the body, as seen by an observer in the B frame. If we now want to express this velocity as seem by an observer in the A frame (i.e., changing the spatial frame), we need to calculate V^h_{ac}. You can do this using the results of part (b) of this problem (which was not assigned), which derived the result:

$$V^h_{ac} = Ad_{-R_{ab}p_{bc}} V^h_{ab} + Ad_{R_{ab}} V^h_{bc} \tag{12}$$

If you chose this approach, then since A and B are stationary, $V^h_{ab} = 0$. Hence, Eq. (12) takes the form:

$$V^h_{ac} = Ad_{R_{ab}} V^h_{bc}$$

Hence, the hybrid velocity is dependent on the orientation of the spatial frame, but not its position.

Alternatively, if you don't want to rely upon part (b), you can recall that the expression for the hybrid velocity is:

$$V^h_{ac} = \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}^s_{ac} \end{bmatrix}$$

Since $\vec{p}_{ac} = \vec{p}_{ab} + R_{ab}\vec{p}_{bc}$, and $\vec{p}_{ab}$ is constant:

$$\dot{\vec{p}}_{ac} = R_{ab}\dot{\vec{p}}_{bc}.$$

Similarly, $\vec{\omega}_{ac} = R_{ab}\vec{\omega}_{bc}$. Hence, V^h_{ac} is dependent of $\vec{p}_{ab}$, but not R_{ab}.

Part (d): Let A be a stationary spatial frame. Let B and C be two different frames attached to a moving body. Let us assume that the velocity of the rigid body is given by V^h_{ab}. If we now switch the location of the body fixed frame from position B to position C, the hybrid velocity of the body is given by V^h_{ac}. Since B and C are both fixed in the body, then $V^h_{bc} = 0$ in Eq. (12). Hence Eq. (12) reduces to:

$$V^h_{ac} = Ad_{-R_{ab}p_{bc}} V^h_{ab}$$

Hence, the hybrid velocity in only dependent on p_{bc}, the position of the body frame, and not on R_{bc}, the orientation of the body fixed frame. Alternatively, you could compute V^h_{ac} in a "brute force" way:

$$\begin{aligned} V^h_{ac} &= \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}(\vec{p}_{ab} + R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ac}R^T_{ac})^\vee \end{bmatrix} = \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}^s_{ab}R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ab}R_{bc}R^T_{bc}R^T_{ab})^\vee \end{bmatrix} \\ &= \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}^s_{ab}R_{ab}\vec{p}_{bc}) \\ \vec{\omega}^s_{ab} \end{bmatrix} = Ad_{-R_{ab}p_{bc}} V^h_{ab} \end{aligned}$$

Thus, the result only depends upon $\vec{p}_{bc}$, and not R_{bc}.