## ME 115(a): Solution to Homework #4

## Problem 1: (15 points)

Each finger applies a "wrench" to the disk object due to its contact with the disk. Since we are assuming a frictionless contact, the finger can only apply forces to the disk that are normal to the disk's boundary. Hence, each finger applies a pure force in the direction of the boundary normal vector, which corresponds to a zero pitch screw.

Define a coordinate system whose origin lies at the common intersection of all of the finger forces at the center of the disk. Choose the z-axis of this system to be normal to the plane of the disk. Let the x-axis coincide with one of the finger contact normals. Thus, the screw coordinates for the three wrenches are:

$$\xi_1 = \begin{bmatrix} -1\\0\\0\\0\\0\\0\\0\end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(120^{\circ})\\-\sin(120^{\circ})\\0\\0\\0\\0\end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(240^{\circ})\\-\sin(240^{\circ})\\0\\0\\0\\0\end{bmatrix}$$

If the disk is not immobilized, there there must exist a twist (i.e., an instantaneous motion of the disk) that is **reciprocal** to the finger wrenches. Let  $\xi_R = [0 \ 0 \ 1 \ 0 \ 0 \ 0]^T$  denote the zero pitch twist that corresponds to rotation of the disk about a vertical axis passing through the origin of the reference frame (i.e., the concurrency point of the three contact normals). This twist is reciprocal to each of the finger wrenches, and therefore the fingers **can not** stop any rotational motions of the disk. Hence, the object is not immobilized.

## Problem 2: (15 points)

**Part (a):** (3 points). Assume that the stationary, or base, frame is chosen to be coincident with the link #1 reference frame when  $\theta_1 = 0$ . Further assume that the tool frame is the same as link frame #3. Then, the Denavit-Hartenberg parameters are:

$$a_0 = 0; \quad \alpha_0 = 0; \quad d_1 = 0; \quad \theta_1 = \text{ variable}$$
  
 $a_1 \neq 0; \quad \alpha_1 = 0; \quad d_2 = 0; \quad \theta_2 = \text{ variable}$   
 $a_2 \neq 0; \quad \alpha_2 = 0; \quad d_3 = \text{ variable}; \quad \theta_3 = 0$ 

You were not asked to derive the forward kinematics of this manipulator. However, in some ways of solving this problem, it is useful to have the forward kinematics:  $g_{S,T} = g_{S,1}g_{1,2}g_{2,3}$ , where,

$$g_{S,1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0\\ \sin\theta_1 & \cos\theta_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} g_{1,2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_1\\ \sin\theta_2 & \cos\theta_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$g_{2,3} = \begin{bmatrix} 1 & 0 & 0 & a_2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d_3\\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{S,3} = \begin{bmatrix} c_{12} & -s_{12} & 0 & (a_1c_1 + a_2c_{12})\\ s_{12} & c_{12} & 0 & (a_1s_1 + a_2s_{12})\\ 0 & 0 & 1 & d_3 \end{bmatrix}$$

where  $c_1 = \cos \theta_1$ ,  $c_{12} = \cos(\theta_1 + \theta_2)$ , etc.

To solve the inverse kinematic equations, one can set the forward kinematic equations equal to the desired tool frame position  $(x_T, y_T, z_T)$ . We immediate see from the forward kinematics that:

$$z_T = d_3.$$

The (x, y) location of the tool frame origin is completely specified by the geometry of the first two links, which look just like the 2R planar manipulator problem that was discussed in class. You could immediately use that solution, or devise a variety of solutions. Here is one approach. Squaring and summing the x and y-coordinates of the tool frame origin yields:

$$x_T^2 + y_T^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\theta_2)$$

which can be solved for  $\theta_2$  (two solutions):

$$\theta_{2} = -\theta_{2}^{'} = \cos^{-1} \left[ \frac{x_{T}^{2} + y_{T}^{2} - a_{1}^{2} - a_{2}^{2}}{2a_{1}a_{2}} \right]$$

There are a variety of ways to determine  $\theta_1$ . Note that from the expressions for the (x, y) tool frame origin we get:

$$x_T - a_1 c_1 = a_2 c_{12} \qquad y_T - a_1 s_1 = a_2 s_{12}$$

Rearranging, we can get two linear equations in the two unknowns  $\cos \theta_1$  and  $\sin \theta_1$ :

$$x_T - (a_1 + a_2 \cos \theta_2) \cos \theta_1 + (a_2 \sin \theta_2) \sin \theta_1 = 0 \tag{1}$$

$$y_T - (a_1 + a_2 \cos \theta_2) \sin \theta_1 - (a_2 \sin \theta_2) \cos \theta_1 = 0$$

$$\tag{2}$$

which can then be solved for each value of  $\theta_2$ .

## Problem 3: (20 Points)

There are different ways to solve this problem. In one approach, we can define the geometry by the Denavit-Hartenberg parameters, and then determine the forward kinematics, from which we can then derive the inverse kinematics by algebraic manipulation. The Denavit-Hartenberg parameters of this manipulator. For simplicity, let us choose the z-axis of the stationary frame to be collinear with the first joint axis. The origin of the stationary frame could be located at any distance along the first axis. However, for simplicity, let's choose the origin of the stationary frame to coincide with the point where the first two joint axes intersect. Also, choose the tool frame origin to coincide with the intersection point of the last three axes (the "wrist"). Also, assume that the link 6 frame which is defined by the Denavit-Hartenberg approach is the tool frame.

$a_0 = 0$	$\alpha_0 = 0$	$d_1 = 0$	$\theta_1 = \text{variable}$
$a_1 = 0$	$\alpha_1 = \frac{\pi}{2}$	$d_2 = 0$	$\theta_2 = \text{variable}$
$a_2 = 0$	$\alpha_2 = -\frac{\pi}{2}$	$d_3 = \text{variable}$	$\theta_3 = 0 \text{ (constant)}$
$a_3 = 0$	$\alpha_3 = \frac{\pi}{2}$	$d_4 = 0$	$\theta_4 = \text{variable}$
$a_4 = 0$	$\alpha_3 = \frac{\pi}{2}$	$d_{5} = 0$	$\theta_5 = \text{variable}$
$a_5 = 0$	$\alpha_3 = \frac{\pi}{2}$	$d_{6} = 0$	$\theta_6 = \text{variable}$
$a_{6} = 0$	$\alpha_3 = \overline{0}$	$d_7 = 0$	$\theta_7 = 0$

**Part (a):** While you were not specifically asked to find the forward kinematics equations, one way to solve the inverse kinematic problem for this manipulator is to construct the forward kinematic equations and then try to find an inverse solution by algebraic manipulation. Using the D-H parameters given above, and the relationship between link frames in terms of the D-H parameters:

$$g_{i,i+1} = \begin{bmatrix} \cos \theta_{i+1} & -\sin \theta_{i+1} & 0 & a_i \\ \sin \theta_{i+1} \cos \alpha_i & \cos \theta_{i+1} \cos \alpha_i & -\sin \alpha_i & -d_{i+1} \sin \alpha_i \\ \sin \theta_{i+1} \sin \alpha_i & \cos \theta_{i+1} \sin \alpha_i & -\cos \alpha_i & d_{i+1} \cos \alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the forward kinematic equations for the first three joints are:

$$g_{ST} = g_{S,1}g_{1,2}g_{2,3}g_{3,T}$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0\\ \sin\theta_1 & \cos\theta_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & -1 & -d_3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2\\ s_1c_2 & c_1 & s_1s_2 & d_3s_1s_2\\ -s_2 & 0 & c_2 & d_3c_2 \end{bmatrix}$$

$$(3)$$

**Inverse Kinematics:** Since  $d_3$  is variable, this manipulator is capable of reaching any desired point  $\vec{p}_D = [x_D \ y_D \ z_D]^T$  within the workspace dictated by the mechanical limits of the joints. Hence, we can find the inverse kinematics by equating terms of  $\vec{p}_{S,3}$  with  $\vec{p}_D$ . First notice that:

$$||\vec{p}_{S,3}||^2 = d_3^2 = (x_D^2 + y_D^2 + z_D^2)$$

Hence,

$$d_3 = \pm (x_D^2 + y_D^2 + z_D^2)^{1/2}$$

We will denote the two possible solutions by  $d_3^+$  and  $d_3^-$ . Next notice that  $d_3 \cos \theta_2 = z_D$  and  $d_3^2 \sin^2 \theta_2 = x_D^2 + y_D^2$ . Hence there are two solutions in  $\theta_2$  for a given choice of  $d_3$ :

$$\theta_2^a = \cos^{-1}[z_D/d_3] \theta_2^b = -\theta_2^a$$

Hence, there are four possible solutions for  $\theta_2$  and  $d_3$ . Finally, we can determine the value of  $\theta_1$  from the x and y components of  $\vec{p}_{S,3}$ :

$$\begin{array}{ll} \cos\theta_1 &= \frac{x_D}{d_3\sin\theta_2} \\ \sin\theta_1 &= \frac{y_D}{d_3\sin\theta_2} \\ \theta_1 &= Atan2[\cos\theta_1,\sin\theta_1] \end{array}$$

There will be one  $\theta_1$  solution for each of the four  $(\theta_2, d_3)$  pairs.

**Part (b):** Here again we will take a similar approach. Based on the Denavit-Hartenberg approach, the forward kinematics of this manipulator has the form:

$$g_{ST}(\theta) = g_{S1}(\theta_1)g_{12}(\theta_2)g_{23}(d_3)g_{34}(\theta_4)g_{45}(\theta_5)g_{56}(\theta_6)g_{6T} .$$

Plugging in the D-H parameters from above yields:

$$g_{ST} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} g_{6T}$$

Because we have already solved for the first three joint variables in part (a) of this problem, it is convenient to express the forward kinematics as the product of these matrices:

$$g_{ST} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2 \\ s_1c_2 & c_1 & s_1s_2 & d_3s_1s_2 \\ -s_2 & 0 & c_2 & d_3c_2 \end{bmatrix} \begin{bmatrix} (c_4c_5c_6 + s_4s_6) & (-c_4c_5s_6 + s_4c_6) & c_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \\ (s_4c_5c_6 - c_4s_6) & -(s_4c_5s_6 + c_4c_6) & s_4s_5 \end{bmatrix} g_{6T}$$
(6)  
$$= g_{S,3} g_{4,6} g_{6T}$$
(7)

The displacement  $g_{6T}$  is a constant (which we assume is the identity matrix with the particular choice of tool frame chosen for this solution, but it may be different for your solution).

For a given choice of  $(\theta_1, \theta_2, \theta_3)$  which was determined in part (a), the remaining variables  $(\theta_4, \theta_5, \theta_6)$  can be solved by rearranging Equation (6):

$$g_{4,6} = g_{S,3}^{-1} g_{ST}^D g_{6T}^{-1} = \begin{bmatrix} R & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix} .$$
(8)

where  $g_{ST}^D$  is the desired tool frame displacement. Letting the elements of the matrix R be denoted by  $r_{ij}$ ,

$$\begin{bmatrix} (c_4c_5c_6 + s_4s_6) & (-c_4c_5s_6 + s_4c_6) & c_4s_5\\ -s_5c_6 & s_5s_6 & c_5\\ (s_4c_5c_6 - c_4s_6) & -(s_4c_5s_6 + c_4c_6) & s_4s_5 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$
(9)

we can then equate terms on the left and right hand sides of this equation to obtain:

$$\cos\theta_5 = r_{23} \; .$$

which has two solutions:

$$\theta_{5,1} = \cos^{-1}(r_{23})$$
 and  $\theta_{5,2} = -\theta_{5,1}$ .

Equating other terms in Equation (9) yields a  $(\theta_4, \theta_5)$ -solution for each value of  $\theta_5$ :

$$\theta_4 = Atan2\left[\frac{r_{13}}{\sin\theta_5}, \frac{r_{33}}{\sin\theta_5}\right]$$
(10)

$$\theta_6 = Atan2[\frac{-r_{21}}{\sin \theta_5}, \frac{r_{22}}{\sin \theta_5}];$$
(11)

Thus, for each of the four  $(\theta_1, \theta_2, \theta_3)$  solutions that place the wrist at the correct spatial location, there are two different wrist solutions.