ME 115(a): Solution to Homework #4
(Winter 2009/2010)

Problem 1: (10 points)

The Euler-Angle representation of the moving body's orientation is

$$R = R_\psi R_\phi R_\gamma$$

where $R_\psi \in SO(3)$ represents rotation by angle ψ about the body fixed z-axis, $R_\phi \in SO(3)$ represents rotation by angle ϕ about the body fixed y-axis, and $R_\gamma \in SO(3)$ represents rotation by angle γ about the body fixed z-axis:

$$R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\phi = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The spatial angular velocity of the rigid body is:

$$\begin{aligned} \hat{\omega}^s &= \dot{R}R^{-1} = (\dot{R}_\psi R_\phi R_\gamma + R_\psi \dot{R}_\phi R_\gamma + R_\psi R_\phi \dot{R}_\gamma)R_\gamma^{-1}R_\phi^{-1}R_\psi^{-1} \\ &= \dot{R}_\psi R_\psi^{-1} + R_\psi(\dot{R}_\phi R_\phi^{-1})R_\psi^{-1} + R_\psi R_\phi(\dot{R}_\gamma R_\gamma^{-1})R_\phi^{-1}R_\psi^{-1} \\ &= \hat{\omega}_\psi^s + R_\psi \hat{\omega}_\phi^s R_\psi^{-1} + R_\psi R_\phi \hat{\omega}_\gamma^s R_\phi^{-1} R_\psi^{-1} \end{aligned} \tag{1}$$

where $\hat{\omega}_\psi^s = \dot{R}_\psi R_\psi^{-1}$, $\hat{\omega}_\phi^s = \dot{R}_\phi R_\phi^{-1}$, $\hat{\omega}_\gamma^s = \dot{R}_\gamma R_\gamma^{-1}$. Converting to vector form:

$$\vec{\omega}^s = (\hat{\omega}^s)^\vee = \vec{\omega}_\psi^s + R_\psi\, \vec{\omega}_\phi^s + R_\psi R_\phi\, \vec{\omega}_\gamma^s \tag{2}$$

The angular velocities are simply:

$$\vec{\omega}_\psi^s = (\dot{R}_\psi R_\psi^{-1})^\vee = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \qquad \vec{\omega}_\phi^s = (\dot{R}_\phi R_\phi^{-1})^\vee = \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} \qquad \vec{\omega}_\gamma^s = (\dot{R}_\gamma R_\gamma^{-1})^\vee \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix}$$

Substituting into Equation (2) results in:

$$\vec{\omega}^s = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} -\dot{\phi}\sin\psi \\ \dot{\phi}\cos\psi \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\gamma}\cos\psi\sin\phi \\ \dot{\gamma}\sin\psi\sin\phi \\ \dot{\gamma}\cos\phi \end{bmatrix} = \begin{bmatrix} -\dot{\phi}\sin\psi + \dot{\gamma}\cos\psi\sin\phi \\ \dot{\phi}\cos\gamma + \dot{\gamma}\sin\psi\sin\phi \\ \dot{\psi} + \dot{\gamma}\cos\phi \end{bmatrix} \tag{3}$$

Problem 2: (15 points) Problem 11(a,b,e) in MLS Chapter 2.

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = \hat{\xi}^3 = \cdots = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}.$$

There are two approaches to compute the exponential for the more general case in which $\omega \neq 0$. One could calculate the expansion in a brute force manner. Alternatively, one could first compute the exponential for a simpler matrix, and then transform the exponential to obtain the desired result. First, let's assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix} \tag{4}$$

be the general twist matrix that we want to exponentiate. Let's transform this matrix to a different coordinate system where the exponentiation will be easier. Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

be the rigid body displacement between the coordinate system in which the twist of Equation (4) is originally defined, and a new coordinate system. In the new coordinate system, the twist, $\hat{\xi}'$, takes the form::

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus, the exponentiation of this twist leads to an algebraically simple formula:

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text, we can transform this result back to the original coordinates to obtain the exponential that we seek:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi' = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi' = (h\hat{\xi}' h^{-1})^\vee \tag{5}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{\xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.$$

Expanding Eq. (5) gives:

$$\xi = (h\hat{\xi}' h^{-1})^\vee = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^\vee = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\hat{\omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\vec{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T\vec{p} + R\vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \vec{v}^s = R\vec{v}^b - R\hat{\omega}^b R^T\vec{p} = R\vec{v}^b - \hat{\omega}^s\vec{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

The spatial translational velocity can be rearranged

$$\vec{v}^s \quad = \quad R\vec{v}^b - \hat{\omega}^s \vec{p} = R\vec{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \tag{6}$$

so that we reach the desired form

$$V^s = \begin{bmatrix} \vec{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix} V^b \ .$$

Problem 3: (10 points) Problem 13 in MLS Chapter 2.

Let ξ_a take the form:

$$\xi_a = \begin{bmatrix} \vec{\rho}_a \times \vec{\omega}_a + h\vec{\omega}_a \\ \vec{\omega}_a \end{bmatrix} \ . \tag{7}$$

Part (a): The configuration of A relative to B is given by g_{ab}^{-1}:

$$g_{ab}^{-1} = \begin{bmatrix} R_{ab}^T & -R_{ab}^T \vec{p}_{ab} \\ \vec{0}^T & 1 \end{bmatrix}$$

Thus, the representation of $\vec{\rho}_a$ and $\vec{\omega}_a$ in B is:

$$\begin{aligned} \vec{\rho}_b &= R_{ab}^T \vec{\rho}_a - R_{ab}^T \vec{p}_{ab} \\ \vec{\omega}_b &= R_{ab}^T \vec{\omega}_a \end{aligned} \tag{8}$$

Substituting these equations into the expression:

$$\begin{aligned} \xi_a &= \begin{bmatrix} \vec{\rho}_b \times \vec{\omega}_b + h\vec{\omega}_b \\ \vec{\omega}_b \end{bmatrix} \\ &= \begin{bmatrix} -R_{ab}^T \hat{\omega}_a \vec{\rho}_a - R_{ab}^T \hat{p}_{ab} \vec{\omega}_a + hR_{ab}^T \vec{\omega}_a \\ R_{ab}^T \vec{\omega}_a \end{bmatrix} \\ &= \begin{bmatrix} R_{ab}^T & -R_{ab}^T \hat{p}_{ab} \\ 0 & R_{ab}^T \end{bmatrix} \begin{bmatrix} (\vec{\rho}_a \times \vec{\omega}_a + h\vec{\omega}_a \\ \vec{\omega}_a \end{bmatrix} \\ &= Ad_{g_{ab}^{-1}} \xi_z \end{aligned} \tag{9}$$

Part (b): The screw is now transformed by a rigid motion $g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix}$. In the new coordinates, the screw location is described by

$$\begin{aligned} \vec{\rho}' &= \vec{p} + R\vec{\rho}_a \\ \vec{\omega}' &= R\vec{\omega}_a \end{aligned}$$

Hence,

$$
\begin{aligned}
\xi' &= \begin{bmatrix} \rho' \times \vec{\omega}' + h\vec{\omega}' \\ \vec{\omega}' \end{bmatrix} = \begin{bmatrix} (\vec{p} + R\vec{\rho}_a) \times (R\vec{\omega}_a) + hR\vec{\omega}_a \\ R\vec{\omega}_a \end{bmatrix} \\
&= \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \vec{\rho}_a \times \vec{\omega}_a + h\vec{\omega}_a \\ \vec{\omega}_a \end{bmatrix} \\
&= Ad_g \xi
\end{aligned} \tag{10}
$$

Problem 4: (10 points) Problem 14 in MLS Chapter 2.

Part (a): Let $g \in SE(3)$ denote a homogeneous transformation matrix and Ad_g the Adjoint transformation associated to displacement g:

$$
g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} .
$$

Then:

$$
g^{-1} = \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_{g^{-1}} = \begin{bmatrix} R^T & -(\widehat{R^T\vec{p}})R^T \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}
$$

where we have made use of the identity $(\widehat{R^T\vec{p}}) = R^T\hat{p}R$. Let's now compute $Ad_g Ad_{g^{-1}}$:

$$
Ad_g Ad_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
$$

Hence, $Ad_{g^{-1}}$ must equal $(Ad_g)^{-1}$ since $Ad_g Ad_{g^{-1}} = I$.

Part (b): If

$$
g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}
$$

Then

$$
g_1 g_2 = \begin{bmatrix} R_1R_2 & \vec{p}_1 + R_1\vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}
$$

Hence:

$$
\begin{aligned}
Ad_{g_1g_2} &= \begin{bmatrix} R_1R_2 & (\vec{p}_1 + R_1\vec{p}_2)\hat{}R_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\
&= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_1^TR_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\
&= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_2 \\ 0 & R_1R_2 \end{bmatrix} \\
&= \begin{bmatrix} R_1 & \hat{p}_1R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2R_2 \\ 0 & R_2 \end{bmatrix} = Ad_{g_1}Ad_{g_2}
\end{aligned}
$$

Problem 5: (15 points) Problem 18(a,b,c,d) in MLS Chapter 2.

Part (a): Let

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & \vec{d}_{ab} \\ \vec{0}^T & 1 \end{bmatrix}$$

denote the relative location of a moving body (with a reference frame "B" attached to the moving body) with respect to a fixed observer in frame "A." The body velocity of the moving body is:

$$\vec{V}^b_{ab} = (g^{-1}_{ab}(t)\dot{g}_{ab}(t))^\vee = \begin{bmatrix} \vec{v}^2_{ab} \\ \vec{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} R^T_{ab}\dot{\vec{d}}_{ab} \\ (R^T_{ab}\dot{R}_{ab})^\vee \end{bmatrix} . \tag{11}$$

To show the desired result,

$$\begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \vec{V}^b_{ab} = \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R^T_{ab}\dot{\vec{d}}_{ab} \\ \vec{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} R_{ab}R^T_{ab}\dot{\vec{d}}_{ab} \\ R_{ab}\vec{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} \dot{\vec{d}}_{ab} \\ \vec{\omega}^s_{ab} \end{bmatrix} = \vec{V}^h_{ab}$$

where we have used the fact that $\vec{\omega}^s_{ab} = R_{ab}\vec{\omega}^b_{ab}$.

Part (b): There are many ways to solve this problem. For example, you could either start with Proposition 2.14 or Proposition 2.15 on page 59 of MLS which relate the velocities of three frames, A, B, and C. Let's choose Prop. 2.15:

$$V^b_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc} \tag{12}$$

Using the fact that

$$V^h_{ac} = \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V^b_{ac}$$

Eq. (12) can be written as:

$$\begin{aligned} V^h_{ac} &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} (Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc}) \\ &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} \begin{bmatrix} R^T_{bc} & -R^T_{bc}\hat{p}_{bc} \\ 0 & R^T_{bc} \end{bmatrix} V^b_{ab} + \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V^b_{bc} \\ &= \begin{bmatrix} R_{ab} & -R_{ab}\hat{p}_{bc} \\ 0 & R_{ab} \end{bmatrix} V^b_{ab} + \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{bc} & 0 \\ 0 & R_{bc} \end{bmatrix} V^b_{bc} \\ &= \begin{bmatrix} I & -\widehat{(R_{ab}p_{bc})} \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} V^b_{ab} + Ad_{R_{ab}} V^h_{bc} \\ &= Ad_{-R_{ab}p_{bc}} V^h_{ab} + Ad_{R_{ab}} V^h_{bc} \end{aligned} \tag{13}$$

Part (c): Let frames A and B be stationary "spatial" frames, and let Frame C be fixed to a moving body. Let V^h_{bc} be the hybrid velocity of the body, as seen by an observer in the B frame. If we now want to express this velocity as seem by an observer in the A frame (i.e., changing the spatial frame), we need to calculate V^h_{ac}. You can do this using the results of part (b) of this problem, which derived the result:

$$V^h_{ac} = Ad_{-R_{ab}p_{bc}} V^h_{ab} + Ad_{R_{ab}} V^h_{bc} \tag{14}$$

If you chose this approach, then since A and B are stationary, $V_{ab}^h = 0$. Hence, Eq. (13) takes the form:

$$V_{ac}^h = Ad_{R_{ab}} V_{bc}^h$$

Hence, the hybrid velocity is dependent on the orientation of the spatial frame, but not its position.

Alternatively, if you don't want to rely upon part (b), you can recall that the expression for the hybrid velocity is:

$$V_{ac}^h = \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac}^s \end{bmatrix}$$

Since $\vec{p}_{ac} = \vec{p}_{ab} + R_{ab}\vec{p}_{bc}$, and $\vec{p}_{ab}$ is constant:

$$\dot{\vec{p}}_{ac} = R_{ab}\dot{\vec{p}}_{bc}.$$

Similarly, $\vec{\omega}_{ac} = R_{ab}\vec{\omega}_{bc}$. Hence, V_{ac}^h is dependent of $\vec{p}_{ab}$, but not R_{ab}.

Part (d): Let A be a stationary spatial frame. Let B and C be two different frames attached to a moving body. Let us assume that the velocity of the rigid body is given by V_{ab}^h. If we now switch the location of the body fixed frame from position B to position C, the hybrid velocity of the body is given by V_{ac}^h. Since B and C are both fixed in the body, then $V_{bc}^h = 0$ in Eq. (13). Hence Eq. (13) reduces to:

$$V_{ac}^h = Ad_{-R_{ab}p_{bc}} V_{ab}^h$$

Hence, the hybrid velocity in only dependent on p_{bc}, the position of the body frame, and not on R_{bc}, the orientation of the body fixed frame. Alternatively, you could compute V_{ac}^h in a "brute force" way:

$$\begin{aligned} V_{ac}^h &= \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}(\vec{p}_{ab} + R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ac}R_{ac}^T)^\vee \end{bmatrix} = \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}_{ab}^s R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ab}R_{bc}R_{bc}^T R_{ab}^T)^\vee \end{bmatrix} \\ &= \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}_{ab}^s R_{ab}\vec{p}_{bc}) \\ \vec{\omega}_{ab}^s \end{bmatrix} = Ad_{-R_{ab}p_{bc}} V_{ab}^h \end{aligned}$$

Thus, the result only depends upon $\vec{p}_{bc}$, and not R_{bc}.

Problem 6: (20 Points) You were asked to calculate the velocity of a moving rigid body assuming that you know the velocities of 3 non-collinear points (P, Q, and R) in the body.

Let points P, Q, and R be located at positions $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ with respect to a fixed observer. Let the velocities of each of these points be $\dot{\vec{r}}_1$, $\dot{\vec{r}}_2$, and $\dot{\vec{r}}_3$. Recall that a spatial velocity has the form

$$\hat{V}^s = \begin{bmatrix} \hat{\omega}^s & \vec{v}^s \\ \vec{0}^T & 0 \end{bmatrix}$$

In general, let $\vec{p}$ denote the location of a particle in the moving body, with respect to the observing frame. Recall that the velocity of this particle, $\dot{\vec{p}}$, as seen in the observing frame, is related to the location of the particle (as seen in the stationary observing frame) by $\dot{\vec{p}} = \hat{V}^s\vec{p}$—where we have used the same notation for homogeneous coordinates. Thus, for each of the particles the following relationship holds:

$$\begin{aligned} \dot{\vec{r}}_1 &= \vec{\omega}^s \times \vec{r}_1 + \vec{v}^s & (15) \\ \dot{\vec{r}}_2 &= \vec{\omega}^s \times \vec{r}_2 + \vec{v}^s & (16) \\ \dot{\vec{r}}_3 &= \vec{\omega}^s \times \vec{r}_3 + \vec{v}^s & (17) \end{aligned}$$

Hence, we have three simultaneous equations in the unknowns $\vec{\omega}^s$ and $\vec{v}^s$. To solve these equations, we follow closely the technique that was used in the handout entitled " Computing the Screw Parameters of a Rigid Body Displacement."

Step 1: subtract Equation 17 from Equations (15) and (16).

$$\begin{aligned} \dot{\vec{r}}_1 - \dot{\vec{r}}_3 &= \vec{\omega}^s \times (\vec{r}_1 - \vec{r}_3) & (18) \\ \dot{\vec{r}}_2 - \dot{\vec{r}}_3 &= \vec{\omega}^s \times (\vec{r}_2 - \vec{r}_3) & (19) \end{aligned}$$

Step 2: Take the cross product of the vector $(\dot{\vec{r}}_2 - \dot{\vec{r}}_3)$ with Equation (18).

$$(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\dot{\vec{r}}_1 - \dot{\vec{r}}_3) = (\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\vec{\omega}^s \times (\vec{r}_1 - \vec{r}_3)). \qquad (20)$$

Step 3: We now use two facts. First, we use the triple vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$. Second, from Equation (19) we know that $\dot{\vec{r}}_2 - \dot{\vec{r}}_3$ is orthogonal to $\vec{\omega}$. Hence, Equation (20) reduces to

$$(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\dot{\vec{r}}_1 - \dot{\vec{r}}_3) = \vec{\omega}[(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \cdot (\vec{r}_1 - \vec{r}_3)].$$

The quantity $\vec{\omega}$ is then easily found as

$$\vec{\omega} = [(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \cdot (\vec{r}_1 - \vec{r}_3)]^{-1}(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \times (\dot{\vec{r}}_1 - \dot{\vec{r}}_3).$$

Step 4: Any one of Equations (15), (16), or (17) could be used to solve for $\vec{v}^s$.