ME 115(a): Solution to Homework #4
(Winter 2013/2014)

Problem 1: (5 points, Prob. 11(e) in Chapt. 2 of MLS)

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\vec{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T \vec{p} + R\vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \vec{v}^s = R\vec{v}^b - R\hat{\omega}^b R^T\vec{p} = R\vec{v}^b - \hat{\omega}^s\vec{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

Using this result:

$$\vec{v}^s = R\vec{v}^b - \hat{\omega}^s\vec{p} = R\vec{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \end{bmatrix} \begin{bmatrix} \vec{v}^b \\ \omega^b \end{bmatrix}$$

Therefore:

$$V^s = \begin{bmatrix} \vec{v}^s \\ \vec{\omega}^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix} V^b$$

Problem 2: (10 points, Problem 14 in Chapter 2 of MLS).

Part (a): Let $g \in SE(3)$ denote a homogeneous transformation matrix:

$$g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Then:

$$g^{-1} = \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_{g^{-1}} = \begin{bmatrix} R^T & -\widehat{(R^T\vec{p})}R^T \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

where we have made use of the identity $\widehat{(R^T\vec{p})} = R^T\hat{p}R$. Let's now compute $Ad_g Ad_{g^{-1}}$:

$$Ad_g Ad_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Hence, $Ad_{g^{-1}}$ must equal $(Ad_g)^{-1}$ since $Ad_g Ad_{g^{-1}} = I$.

Part (b): If

$$g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Then

$$g_1 g_2 = \begin{bmatrix} R_1R_2 & \vec{p}_1 + R_1\vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence:

$$\begin{aligned} Ad_{g_1g_2} &= \begin{bmatrix} R_1R_2 & (\vec{p}_1 + R_1\vec{p}_2)\hat{}R_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_1^TR_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 & \hat{p}_1R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2R_2 \\ 0 & R_2 \end{bmatrix} = Ad_{g_1}Ad_{g_2} \end{aligned}$$

Problem 3: (25 points, Problem 18(a,b,c,d,e) in Chapter 2 of MLS).

Part (a): Let

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & \vec{d}_{ab} \\ \vec{0}^T & 1 \end{bmatrix}$$

denote the relative location of a moving body (with a reference frame "B" attached to the moving body) with respect to a fixed observer in frame "A." The body velocity of the moving body is:

$$\vec{V}^b_{ab} = (g^{-1}_{ab}(t)\dot{g}_{ab}(t))^\vee = \begin{bmatrix} \vec{v}^2_{ab} \\ \vec{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} R^T_{ab}\dot{\vec{d}}_{ab} \\ (R^T_{ab}\dot{R}_{ab})^\vee \end{bmatrix} . \tag{1}$$

To show the desired result,

$$\begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \vec{V}^b_{ab} = \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R^T_{ab}\dot{\vec{d}}_{ab} \\ \vec{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} R_{ab}R^T_{ab}\dot{\vec{d}}_{ab} \\ R_{ab}\vec{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} \dot{\vec{d}}_{ab} \\ \vec{\omega}^s_{ab} \end{bmatrix} = \vec{V}^h_{ab}$$

where we have used the fact that $\vec{\omega}^s_{ab} = R_{ab}\vec{\omega}^b_{ab}$.

Part (b): There are many ways to solve this problem. For example, you could either start with Proposition 2.14 or Proposition 2.15 on page 59 of MLS which relate the velocities of three frames, A, B, and C. Let's choose Prop. 2.15:

$$V^b_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc} \tag{2}$$

Using the fact that

$$V_{ac}^h = \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V_{ac}^b$$

Eq. (2) can be written as:

$$\begin{aligned} V_{ac}^h &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} (Ad_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b) \\ &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} \begin{bmatrix} R_{bc}^T & -R_{bc}^T \hat{p}_{bc} \\ 0 & R_{bc}^T \end{bmatrix} V_{ab}^b + \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V_{bc}^b \\ &= \begin{bmatrix} R_{ab} & -R_{ab} \hat{p}_{bc} \\ 0 & R_{ab} \end{bmatrix} V_{ab}^b + \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{bc} & 0 \\ 0 & R_{bc} \end{bmatrix} V_{bc}^b \\ &= \begin{bmatrix} I & -(\widehat{R_{ab} p_{bc}}) \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} V_{ab}^b + Ad_{R_{ab}} V_{bc}^h \\ &= Ad_{-R_{ab} p_{bc}} V_{ab}^h + Ad_{R_{ab}} V_{bc}^h \end{aligned} \tag{3}$$

Part (c): Let frames A and B be stationary "spatial" frames, and let Frame C be fixed to a moving body. Let V_{bc}^h be the hybrid velocity of the body, as seen by an observer in the B frame. If we now want to express this velocity as seem by an observer in the A frame (i.e., changing the spatial frame), we need to calculate V_{ac}^h. You can do this using the results of part (b) of this problem, which derived the result:

$$V_{ac}^h = Ad_{-R_{ab} p_{bc}} V_{ab}^h + Ad_{R_{ab}} V_{bc}^h \tag{4}$$

If you chose this approach, then since A and B are stationary, $V_{ab}^h = 0$. Hence, Eq. (3) takes the form:

$$V_{ac}^h = Ad_{R_{ab}} V_{bc}^h$$

Hence, the hybrid velocity is dependent on the orientation of the spatial frame, but not its position.

Alternatively, if you don't want to rely upon part (b), you can recall that the expression for the hybrid velocity is:

$$V_{ac}^h = \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac}^s \end{bmatrix}$$

Since $\vec{p}_{ac} = \vec{p}_{ab} + R_{ab} \vec{p}_{bc}$, and $\vec{p}_{ab}$ is constant:

$$\dot{\vec{p}}_{ac} = R_{ab} \dot{\vec{p}}_{bc}.$$

Similarly, $\vec{\omega}_{ac} = R_{ab} \vec{\omega}_{bc}$. Hence, V_{ac}^h is dependent of $\vec{p}_{ab}$, but not R_{ab}.

Part (d): Let A be a stationary spatial frame. Let B and C be two different frames attached to a moving body. Let us assume that the velocity of the rigid body is given by V_{ab}^h. If we now switch the location of the body fixed frame from position B to position C, the hybrid velocity of the body is given by V^h_{ac}. Since B and C are both fixed in the body, then $V^h_{bc} = 0$ in Eq. (3). Hence Eq. (3) reduces to:

$$V^h_{ac} = Ad_{-R_{ab}p_{bc}} V^h_{ab}$$

Hence, the hybrid velocity in only dependent on p_{bc}, the position of the body frame, and not on R_{bc}, the orientation of the body fixed frame. Alternatively, you could compute V^h_{ac} in a "brute force" way:

$$\begin{aligned} V^h_{ac} &= \begin{bmatrix} \dot{\vec{p}}_{ac} \\ \vec{\omega}_{ac} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}(\vec{p}_{ab} + R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ac}R^T_{ac})^\vee \end{bmatrix} = \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}^s_{ab}R_{ab}\vec{p}_{bc}) \\ (\dot{R}_{ab}R_{bc}R^T_{bc}R^T_{ab})^\vee \end{bmatrix} \\ &= \begin{bmatrix} \dot{\vec{p}}_{ab} + \hat{\omega}^s_{ab}R_{ab}\vec{p}_{bc}) \\ \vec{\omega}^s_{ab} \end{bmatrix} = Ad_{-R_{ab}p_{bc}} V^h_{ab} \end{aligned}$$

Thus, the result only depends upon $\vec{p}_{bc}$, and not R_{bc}.

Part (e): Let the position and orientation of a moving rigid body be given by $R(t)$ and $\vec{p}(t)$. Let V^b be the body velocity of the rigid body, and let F^b be a wrench applied to the body, expressed in body coordinates. The power applied to the body due to this wrench is given by:

$$V^b \cdot F^b = (V^b)^T F^b \tag{5}$$

Let V^h denote the velocity of the body in hybrid coordinates. Similarly, define the hybrid wrench to be F^h. We will define F^h to be the wrench that preserves the amount of power in Eq. (5):

$$\begin{aligned} V^b \cdot F^b = (V^b)^T F^b &= V^h \cdot F^h \\ &= (V^h)^T F^h \\ &= (\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} V^b)^T F^h \\ &= (V^b)^T \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h \end{aligned}$$

Hence, it must be true that:

$$F^b = \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h \quad \text{or} \quad F^h = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} F^b$$

Problem 4: (10 points, Problem 15 in Chapter 2 of MLS)

Since $g_{ac} = g_{ab}g_{bc}$, using the product rule for differentiation yields:

$$\begin{aligned} \hat{V}^b_{ac} &= g^{-1}_{ac}\dot{g}_{ac} \\ &= g^{-1}_{bc}g^{-1}ab(\dot{g}_{ab}g_{bc} + g_{ab}\dot{g}_{bc}) \\ &= g^{-1}_{bc}\hat{V}^b_{ab}g_{bc} + \hat{V}^b_{bc} \end{aligned} \tag{6}$$

Converting to vector form via the $\vee$ operator yields:

$$\vec{V}_{ac}^b = (\hat{V}_{ac}^b)^\vee = Ad_{g_{bc}^{-1}} \vec{V}_{ab}^c + \vec{V}_{bc}^b$$

Problem 5: (10 Points, Problem 16(a,b) in Chapter 2 of MLS)

Part (a): $g_{0,3}$ can be determined in a variety of ways, such as by using the Denavit-Hartenberg, the product of exponentials (POE) approach, or a "brute force" approach. Let's use the POE. Assume that the reference configuration is that given in Figure 2.17 of MLS. Hence, $g_{ST}(0)$ is:

$$g_{ST}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (l_1 + l_2) \\ 0 & 0 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The twist coordinates of the joint axes (in the reference configuration) are:

$$\vec{\xi}_1 = \begin{bmatrix} h_1\vec{\omega}_1 + \rho_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{\xi}_1 = \begin{bmatrix} h_1\vec{\omega}_2 + \rho_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The forward kinematics is then given by

$$\begin{aligned} g_{ST} &= e^{\theta_1\hat{\xi}_1} e^{\theta_2\hat{\xi}_2} g_{ST}(0) \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & -(l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2)) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{7}$$

Part (b): Given g_{ST}, the spatial velocity can easily be computed as

$$\vec{V}_{ST}^s = (\dot{g}_{ST} g_{ST}^{-1})^\vee. \tag{8}$$

Later will will learn that one can formally rearrange these equations into the form:

$$\vec{V}_{ST}^s = J_{ST}^s \dot{\vec{\theta}}$$

where J_{ST}^s is termed the *spatial Jacobian matrix*. One could substitute Eq. (7) directly into Eq. (8) and carry through with the tedious algebra. To get a "hint" about the Jacobian matrix, note that

$$
\begin{aligned}
\dot{g}_{ST} g_{ST}^{-1} &= \frac{d}{dt}\left(e^{\theta_1\hat{\xi}_1} e^{\theta_1\hat{\xi}_2} g_{ST}(0)\right)\left(e^{\theta_1\hat{\xi}_1} e^{\theta_2\hat{\xi}_2} g_{ST}(0)\right)^{-1} \\
&= \left(\dot{\theta}_1\hat{\xi}_1 e^{\theta_1\hat{\xi}_1} e^{\theta_1\hat{\xi}_2} g_{ST}(0) + e^{\theta_1\hat{\xi}_1}\dot{\theta}_2\hat{\xi}_2 e^{\theta_2\hat{\xi}_2} g_{ST}(0)\right) g_{ST}^{-1}(0) e^{-\theta_2\hat{\xi}_2} e^{-\theta_1\hat{\xi}_1} \\
&= \dot{\theta}_1\hat{\xi}_1 + \dot{\theta}_2 e^{\theta_1\hat{\xi}_1}\hat{\xi}_2 e^{-\theta_1\hat{\xi}_1} \qquad (9)
\end{aligned}
$$

Hence, the spatial Jacobian matrix takes the form:

$$
\begin{aligned}
J_{ST}^s &= \begin{bmatrix} \vec{\xi}_1 & \vec{\xi}_2' \end{bmatrix} = \begin{bmatrix} \vec{\xi}_1 & Ad_{e^{\theta_1\hat{\xi}_1}}\vec{\xi}_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & l_1\cos\theta_1 \\ 0 & l_1\sin\theta_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

Part (c): The body velocity can be computed as a function of the body Jacobian matrix, or can be computed as the adjoint of the spatial velocity found in part (b). In either case, the result is:

$$
\vec{V}_{ST}^b = J_{ST}^b\dot{\vec{\theta}} = \begin{bmatrix} -(l_2 + l_1\cos\theta_2 & -l_2 \\ l_1\sin\theta_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$