ME 115(a): Solution to Homework #5

(Winter 2009/2010)

Problem 1:

Each finger applies a "wrench" to the disk object due to its contact with the disk. Since we are assuming a frictionless contact, the finger can only apply forces to the disk that are normal to the disk's boundary. Hence, each finger applies a pure force in the direction of the boundary normal vector, which corresponds to a zero pitch screw.

Define a coordinate system whose origin lies at the common intersection of all of the finger forces at the center of the disk. Choose the z-axis of this system to be normal to the plane of the disk. Let the x-axis coincide with one of the finger contact normals. Thus, the screw coordinates for the three wrenches are:

$$\xi_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(120^o) \\ -\sin(120^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(240^o) \\ -\sin(240^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If the disk is not immobilized, there there must exist a twist (i.e., an instantaneous motion of the disk) that is **reciprocal** to the finger wrenches. Let $\xi_R = [0\ 0\ 1\ 0\ 0\ 0]^T$ denote the zero pitch twist that corresponds to rotation of the disk about a vertical axis passing through the origin of the reference frame (i.e., the concurrency point of the three contact normals). This twist is reciprocal to each of the finger wrenches, and therefore the fingers **can not** stop any rotational motions of the disk. Hence, the object is not immobilized.

Problem 2: (Problem 2.18(e) in MLS text)

Part (e): Let the position and orientation of a moving rigid body be given by $R(t)$ and $\vec{p}(t)$. Let V^b be the body velocity of the rigid body, and let F^b be a wrench applied to the body, expressed in body coordinates. The power applied to the body due to this wrench is given by:

$$V^b \cdot F^b = (V^b)^T F^b \tag{1}$$

Let V^h denote the velocity of the body in hybrid coordinates. Similarly, define the hybrid wrench to be F^h. We will define F^h to be the wrench that preserves the amount of power in Eq. (1):

$$V^b \cdot F^b = (V^b)^T F^b \begin{aligned}[t] &= V^h \cdot F^h \\ &= (V^h)^T F^h \\ &= (\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} V^b)^T F^h \\ &= (V^b)^T \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h \end{aligned}$$

Hence, it must be true that:

$$F^b = \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} F^h \qquad \text{or} \qquad F^h = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} F^b$$

Problem 3: (Problem 2.16(a,b,c) in MLS text)

Part (a): $g_{0,3}$ can be determined using the Denavit-Hartenberg approach or the product of exponentials (POE) approach. Let's use the POE. Assume that the reference configuration is that given in Figure 2.17 of MLS. Hence, $g_{ST}(0)$ is:

$$g_{ST}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (l_1 + l_2) \\ 0 & 0 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The twist coordinates of the joint axes (in the reference configuration) are:

$$\vec{\xi}_1 = \begin{bmatrix} h_1 \vec{\omega}_1 + \rho_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{\xi}_1 = \begin{bmatrix} h_1 \vec{\omega}_2 + \rho_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The forward kinematics is then given by

$$\begin{aligned} g_{ST} &= e^{\theta_1 \hat{\xi}_1} e^{\theta_2 \hat{\xi}_2} g_{ST}(0) \\ &= \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 & -(l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 & l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Part (b): The spatial velocity can be computed as a function of the spatial Jacobian matrix:

$$\vec{V}^s_{ST} = J^s_{ST} \dot{\vec{\theta}}$$

where:

$$J^s_{ST} = \begin{bmatrix} \vec{\xi}_1 & \vec{\xi}_2' \end{bmatrix} = \begin{bmatrix} \vec{\xi}_1 & Ad_{e^{\theta_1 \hat{\xi}_1}} \vec{\xi}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & l_1 \cos\theta_1 \\ 0 & l_1 \sin\theta_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Part (c): The body velocity can be computed as a function of the body Jacobian matrix, or can be computed as the adjoint of the spatial velocity found in part (b). In either case, the result is:

$$\vec{V}^b_{ST} = J^b_{ST}\dot{\vec{\theta}} = \begin{bmatrix} -(l_2 + l_1\cos\theta_2 & -l_2 \\ l_1 \sin\theta_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Problem 4: *Consider the two screws, S_1 and S_2, shown in Figure 1(a). The screw axis of S_1 is perpendicular to the plane, P, and has zero pitch: $h_1 = 0$. The screw axis of S_2 lies in P, and S_2 some non-zero pitch, h_2. The distance between S_1 and S_2, as measured along a mutually perpendicular line, is denoted a. Describe the set of all screws whose axes lie in P and that are reciprocal to both S_1 and S_2.*

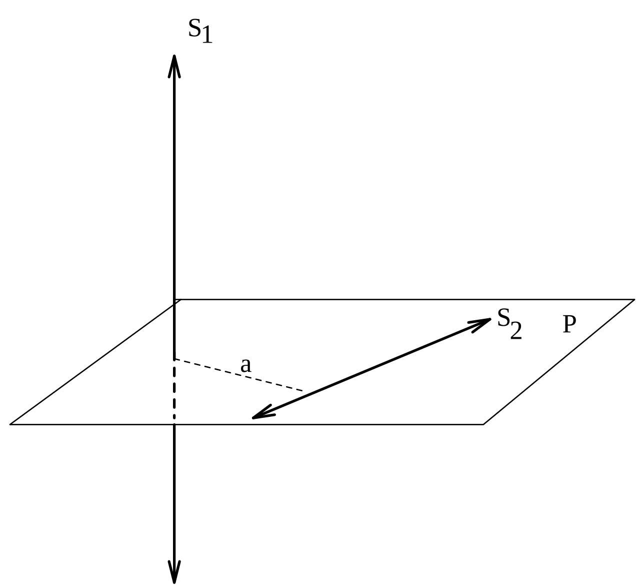

Figure 1: Two Screws.

Assign a coordinate system whose origin is located at a point C, where the screw axis of S_1 intersects the horizontal plane, P, containing S_2. Let the z-axis be collinear with the positive S_1 direction, and let the x-axis be collinear with the mutually perpendicular line between S_1 and S_2. In this coordinate system, the screw coordinates of S_1 and S_2 are:

$$\xi_1 = \begin{bmatrix} h_1\vec{\omega}_1 + \vec{\rho}_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} h_2\vec{\omega}_2 + \vec{\rho}_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ h_2 \\ a \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We require that any screw, S_R, which is reciprocal to both S_1 and S_2 also lie in the plane P. We can parametrize all screws that lie in P as follows:

$$\xi_R = \begin{bmatrix} h_R\vec{\omega}_R + \vec{\rho}_R \times \vec{\omega}_R \end{bmatrix}$$

where h_R is the pitch of the reciprocal screw while $\vec{\omega}_R$ is a unit length vector collinear with the screw axis of the reciprocal screw, $\vec{\rho}_R$ is a vector from the origin of the reference frame described above to a point on the reciprocal screw axis. By assumption, both $\vec{\omega}_R$ and $\vec{\rho}_R$ must also lie in P. We can describe any screw that lies in the plane by two scalars: d (the distance along the mutually perpendicular line between S_1 and S_R) and θ, the angle between the mutuallyperpendicular line and the x-axis of the reference coordinate system, which lies in P. In terms of these scalars:

$$\vec{\omega}_R = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad \vec{\rho}_R = d \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and hence:

$$\xi_R = \begin{bmatrix} -h_R\sin\theta \\ h_R\cos\theta \\ d \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

If S_R is reciprocal to S_1, then the reciprocal product between these two screws must be zero. Letting $\circ$ denote the reciprocal product,

$$\xi_1 \circ \xi_R = d = 0$$

This implies that the screw axis of S_R must intersect the axis of S_1. If S_R is reciprocal to S_2, then:

$$\xi_2 \circ \xi_R = (h_2 + h_R)\cos\theta = 0.$$

Hence, S_R must always intersect S_1 and either have the negative pitch of S_2, or it can have any pitch if it is orthogonal to the axis of S_2 (i.e. $\cos\theta = 0$).