ME 115(b): Homework #5 Soltution

Problem #1:

Recall that the definition of the evolute, $\beta(t)$, of a *planar* curve, $\alpha(t)$, is:

$$\beta(t) = \alpha(t) + \frac{1}{\kappa(t)}\vec{n}(t)$$

where $\kappa(t)$ is the curvature of $\alpha(t)$ and $\vec{n}(t)$ is the unit normal vector of $\alpha(t)$ at t. Let's assume that the curve is arc-length parametrized, i.e., t is the arc-length parameter.

The tangent to the evolute is simply derived by taking the derivative of the evolute equation.

$$\frac{d\beta}{dt} = \vec{u} + \frac{d}{dt}\left(\frac{1}{\kappa(t)}\vec{n}(t)\right) = \vec{u} + \frac{\kappa(t)\vec{n}^{'}(t) - \vec{n}(t)\kappa^{'}(t)}{\kappa^2(t)} \tag{1}$$

Recall that for a regular curve, $\vec{u}(t) = \alpha^{'}(t)$, $\vec{u}^{'}(t) = \kappa(t)\vec{n}(t)$.

What is $\vec{n}^{'}(t)$ for a planar curve? Since $\vec{n}(t)$ is a unit length curve, then $\vec{n}^{'}(t)$ must be a vector orthogonal to $\vec{n}(t)$–i.e., a vector in the direction of $\vec{u}(t)$. Thus, assume that $\vec{n}^{'} = \gamma\vec{u}$, where γ is some proportionality constant, which is to be determined.

From the relationship $\vec{u} \cdot \vec{n} = 0$, we can obtain (by taking derivatives):

$$\begin{aligned} 0 &= \vec{u}^{'} \cdot \vec{n} + \vec{u} \cdot \vec{n}^{'} && (2)\\ &= \kappa + \vec{u} \cdot \vec{n}^{'} && (3)\\ &= \kappa + \vec{u} \cdot (\gamma\vec{u}) && (4)\\ &= \kappa + \gamma && (5) \end{aligned}$$

This implies that:

$$\vec{n}^{'} = -\kappa\vec{u}$$

Combining these results, gives:

$$\begin{aligned} \frac{d\beta(t)}{dt} &= \vec{u} + \frac{\kappa(t)\vec{n}^{'} - \vec{n}(t)\kappa^{'}}{\kappa^2} && (6)\\ &= \vec{u} + \frac{-\kappa^2\vec{u} - \kappa^{'}\vec{n}}{\kappa^2} && (7)\\ &= \frac{-\kappa^{'}}{\kappa}\vec{n}r && (8) \end{aligned}$$

Thus, $d\beta/dt$ is a vector in the direction of the normal to $\alpha(t)$.

Part (b): The evolute of a circle is simply a point.

For a circle:

$$\alpha(t) = \begin{bmatrix} r\cos(\frac{t}{r}) \\ r\sin(\frac{t}{r}) \end{bmatrix} \tag{9}$$

$$\beta(t) = \begin{bmatrix} r\cos(\frac{t}{r}) \\ r\sin(\frac{t}{r}) \end{bmatrix} + r\begin{bmatrix} -\cos(\frac{t}{r}) \\ sin(\frac{t}{r}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

Problem #2

Part (a): Let's define the object frames on the two bodies as follows:

Recall that the boundary of the ellipse can thus be parametrized as:

$$c_1(\mu_1) = \begin{bmatrix} a\cos(\mu_1) \\ b\sin(\mu_1) \end{bmatrix}$$

where μ_1 is the "curve parameter" of the ellipse boundary. It is not necessarily the arc-length parameter. Note, with this parametrization, the normal vector (see below) is pointing *inward* into the object. The circle can be paramterized as:

$$c_2(\mu_2) = \begin{bmatrix} r\cos(\mu_2) \\ -r\sin(\mu_2) \end{bmatrix}$$

Thus, the tangent vectors (not necessarily unit length) to the two surfaces are:

$$\vec{t}_1(\mu_1) = \begin{bmatrix} -a\sin(\mu_1) \\ b\cos(\mu_1) \end{bmatrix} \tag{11}$$

$$\vec{t}_2(\mu_2) = \begin{bmatrix} -r\sin(\mu_2) \\ -r\cos(\mu_2) \end{bmatrix} \tag{12}$$

Similarly:

$$M_1 = |\vec{t}_1| = (a^2\sin^2\mu_1 + b^2\cos^2\mu_1)^{1/2} \tag{13}$$

$$M_2 = |\vec{t}_2| = r \tag{14}$$

Recall that the curvature of the i^{th} planar surface could be computed as:

$$\kappa_i = -M_i^{-2}(\frac{\partial c_i}{\mu_i})^T \frac{\partial \vec{n}_i}{\partial \mu_i}$$

For planar curves, the normal can be easily computed as the unit tangent vector rotated by $\pm\pi/2$

$$\vec{n}_1 = -\frac{1}{(a^2\sin^2\mu_1 + b^2\cos^2\mu_1)^{1/2}}\begin{bmatrix} b\cos\mu_1 \\ a\sin\mu_1 \end{bmatrix}$$

Consequently:

$$\frac{\partial \vec{n}_i}{\partial \mu_1} = -\frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} \begin{bmatrix} -a \sin \mu_1 \\ b \cos \mu_1 \end{bmatrix}$$

Thus:

$$\kappa_1 = \frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}}$$

$$\vec{n}_2 = \begin{bmatrix} -r \cos(\mu_2) \\ -r \sin(\mu_2) \end{bmatrix}$$

Consequently:

$$\frac{\partial \vec{n}_2}{\partial \mu_2} = R$$

Thus:

$$\kappa_2 = \frac{1}{r}$$

Assuming that v_t and $\dot{\theta}$ are given, the contact equations are:

$$\begin{aligned}
\dot{\mu_1} &= (\kappa_1 + \kappa_2)^{-1} {M_1}^{-1} (-\dot{\theta} + \kappa_2 v_t) && (15) \\
&= \left(\frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} + \frac{1}{r}\right)^{-1} (a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{1/2-1} (-\dot{\theta} + \frac{1}{r} v_t) && (16) \\
\dot{\mu_2} &= (\kappa_1 + \kappa_2)^{-1} {M_2}^{-1} (\dot{\theta} + \kappa_1 v_t) && (17) \\
&= \left(\frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} + \frac{1}{r}\right)^{-1} \frac{1}{r} (\dot{\theta} + \frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} v_t) && (18)
\end{aligned}$$

The relative curvature is ill defined when $\kappa_1 + \kappa_2 = 0$. In other words, when:

$$\frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{\frac{3}{2}}} = -\frac{1}{r} \quad (19)$$