CDS 110(b) Final Exam

(Winter 2011/2012)

Instructions

1. Limit your total time to 5 hours. It is okay to take a break in the middle of the exam if you need to ask the Instructor or TA a question, or to go to dinner. If you run out of time, indicate how you would proceed as explicitly as possible.
2. You may use any class notes, books, or other written material posted on the course web site. You may not discuss this final with other class students or other people except me or the class Teaching Assistants.
3. You may use Mathematica, MATLAB, or any software or computational tools to assist you.
4. You cannot use the internet to solve these problems, except for material on the course web site.
5. The final is due by 5:00 p.m. on the last day of finals.
6. The point values are listed for each problem to assist you in allocation of your time.

Problem 1: The goal of preserving fuel or energy use is critical for spacecraft design and deployment. In this problem you will consider a highly simplified version of a satellite attitude control problem. Assume a single rigid body satellite free=floating above earth is constrained to move in a plane. In this simplified model, θ is the angle which describes the satellite. The dynamics which relate the control input, u to the satellite's attitude are:

$$I\ddot{\theta} \;=\; u \tag{1}$$

where I is the rotational inertia of the satellite, and u can be interpreted as the torque applied to the satellite. In practice, the torque can be provided by an *inertia wheel* or by a *thruster*.

Assume that the spacecraft must carry out a reorientation maneuver, starting from an initial orientation θ_0 at time $t_0 = 0$, and ending at a final orientation, θ_f, at $t_f = T$. Also assume that the control is bounded: $|u| \leq 1$.

Part (a): (5 points) If a reaction wheel is used to provide the reorienting maneuver, then the cost of reorientation can be modeled as

$$J(u) \;=\; \int_0^T (1 \;+\; \alpha\, u^2)dt \tag{2}$$

where $\alpha \geq 0$ is a weighting factor that trades off maneuver time ($\alpha = 0$ gives the minimum time solution, while $\alpha \to \infty$ gives the minimum fuel solution). First, convert this system to state space form. For convenience, you can assume that the rotational inertia takes unit value: $I = 1$. Write down the necessary conditions for optimality using Pontryagin's maximum principle, including boundary conditions.

Part(b): (5 points) *Sketch* the time-history of the control, u, reorientation rate, $\dot{\theta}$, and orientation, θ, during a reorientation maneuver. You *do not* need to solve these equations explicitly. Instead, sketch the qualitative behavior of the solution. You may want to consider three cases: (1) minimum time ($\alpha = 0$); (2) balance between minimum time and minmum, with $\alpha = 1$, and (3) high cost energy, with $\alpha = 10$.

Part (c): (10 points) When a thruster is used (e.g., a compressed gas canister, whose loss of gas is roughly proportional to the amount of thrust generated), one can approximately model the fuel use as:

$$J(u) \;=\; \int_0^T (1 \;+\; \alpha\, |u|)dt \tag{3}$$

Show that the optimal control solution for this cost is a bang-deadzone-bang solution, and write the switching conditions.

Part (d): (5 points) Sketch qualitatively the solution of part (c) for a nonzero value of α, e.g., $\alpha = 1$.

Problem 2: Consider the 1-dimensional second order system

$$m\ddot{z} + kz \;= u + \tilde{\eta} \tag{4}$$

where z is the system state, u is the control input, and $\tilde{\eta}$ is white, zero mean, Gaussian noise. Taylor series expansion can be used to discretize the system as follows

$$\begin{bmatrix} z_{k+1} \\ \dot{z}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \delta t \\ -k\delta t/m & 1 \end{bmatrix} \begin{bmatrix} z_k \\ \dot{z}_k \end{bmatrix} + \begin{bmatrix} 0 \\ \delta t/m \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \eta_k \tag{5}$$

where η_k is white, zero mean, Gaussian noise with covariance Q_k, and δt is the sampling time of the model discretization. Also assume that the position, z, is measured with a noisy sensor, so that

$$y_k = z_k + \omega_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_k \\ \dot{z}_k \end{bmatrix} + \omega_k \tag{6}$$

where ω_k is a zero mean, white, Gaussian noise with covariance R_k.

Part (a): (5 points) The goal is to estimate both the position *and* velocity of the moving system. Assuming $m = 1$, $k = 1$ and $\delta t = 1$, solve for the steady state Kalman Filter gains analytically.

Part (b): (5 points) Using the same parameter assumptions as above, plot the closed loop estimator pole locations when the *Signal-to-Noise Ratio* $\sqrt{Q_k/R_k}$ takes values of 0.1, 1.0, and 10.0. What is the difference

Part (c): (10 points) During the course, we studied in detail the *fixed lag smoother*. We briefly defined the *fixed point smoother*, but did not study it in detail. In this part of the problem you are to develop a fixed point smoother for the initial state of the dynamical system described in this problem. Assume that at time t_0 the initial state of the system is an uncertain variable with mean and covariance:

$$E\begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix} = \begin{bmatrix} \bar{z}_0 \\ \bar{v}_0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.25 \end{bmatrix} \qquad P_0 = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.5 \end{bmatrix} . \tag{7}$$

Recall that a fixed point smoother computes the estimate $\hat{x}_{j|k}$ of a state x at times $k = j, j+1, j+2, \ldots$. Assuming that $u = 0$, compute the estimates of the initial position, $\hat{z}_{0|k}$, and initial velocity, $\hat{v}_{0|k}$, for $k = 0, 1, 2, 3, 4, 5$, and a signal-to-noise ratio of 1.