CDS 110(b): Winter 2011/2012

(due Wednesday, Feb 1, 2012) [1]

Problem #1: Consider the following LQR problem for a problem with two states, $\vec{x} = [x_1 \; x_2]^T$. The system is governed by the dynamical equations:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \vec{x} \; + \; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} b & 1 \end{bmatrix} \tag{2}$$

where u is a single scalar control input. The performance criteria is defined by the following infinite horizon cost:

$$V = \int_0^\infty (x_1^2 \; + \; u^2) \; dt \tag{3}$$

Let P have the form:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{4}$$

where $p_{12} = p_{21}$ and $P > 0$ since P must be positive definite symmetric.

(a) Write the steady state Riccati equation and reduce it to equations in the elements of P and the constants.

(b) Find the gains for the optimal controller, assuming full state feedback.

(c) Find the closed loop natural frequency and damping ratio.

Problem #2: Derive the LQR solution for a finite-horizon discrete-time optimal control problem using Bellman's equation. That is, given the time-varying value cost:

$$V(x,k) \;=\; \sum_{i=k}^{N-1} (x_i^T Q x_i \;+\; u_i^T R u_i) \;+\; x_N^T Q_N x_N \tag{5}$$

derive the feedback law $u_k \;=\; K_k x_k$ that minimizes this cost. Start with Bellman's equation, adapted to the discrete time case:

$$V(x_k,k) \;=\; \min_u \; [L(x,k) \;+\; V(x_{k+1}, k+1)]$$

and assume that $V(x_k,k) \;=\; x_k^T P_k x_k$.

[1] Please put the number of hours you spent on this homework (including reading) in the upper left-hand corner of the first page of your homework